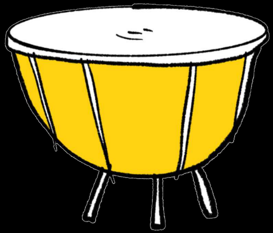


# Geometry emerging from spectra


Walter van Suijlekom

Radboud Universiteit

# Lorentz in October 1910

*H.A. Lorentz by Jan Veth*

Origins of spectral geometry:

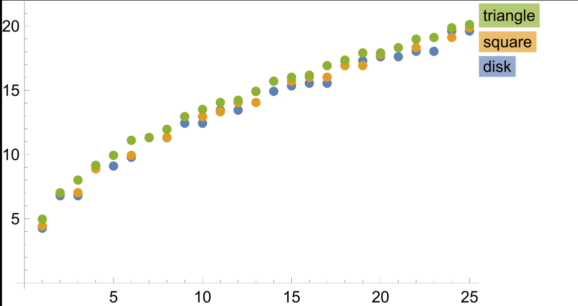
*the high overtones behave inversely proportional to the volume.*

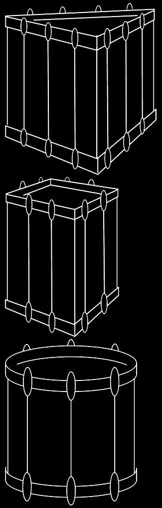
# Weyl in February 1911

$$N(\Lambda) = \#\text{wave numbers} \leq \Lambda$$

$$\sim \frac{\Omega_d \mathrm{Vol}(M)}{d(2\pi)^d} \Lambda^d$$

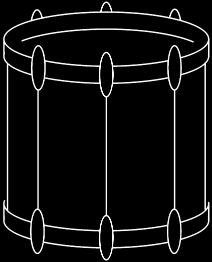
Evidence by the parabolic shapes ($\sqrt{\Lambda}$):



# Mark Kac in 1966

"Can one hear the shape of a drum?"

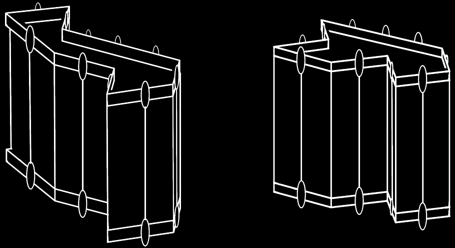


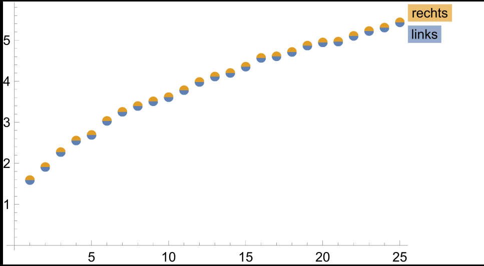
Or, more precisely, given a Riemannian manifold $M$, does the spectrum of wave numbers $k$ in the Helmholtz equation

$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of $M$?

# *Isospectral* drums!





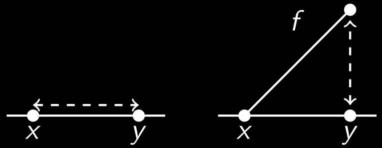

… so the answer to Kac's question is **no**
and more information is needed…

# Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

- Distance $d(x, y)$ between two points is usually defined as
  *the* **smallest** *of the arclengths (computed using the metric) of curves connecting $x$ and $y$.*
- But it can also be defined as
  *the* **largest** *of differences $|f(x) - f(y)|$ for functions $f$ with gradient $|\nabla f| \leq 1$.*

$$d(x, y) = \sup_{\|[D_M, f]\| \leq 1} |\delta_x(f) - \delta_y(f)|$$

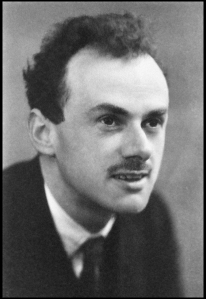



Combination $(C^\infty(M), L^2(S_M), D_M)$
allows for reconstruction of geometry

# Analysis: Dirac operator

Recall that $k^2$ is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator is a 'square-root' of the Laplacian, so that its spectrum give the wave numbers $k$.
- First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold $M$.



# The circle

- The Laplacian on the circle $\mathbb{S}^1$ is given by

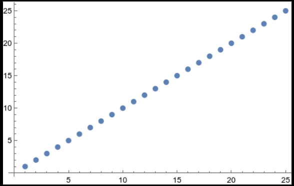
$$\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}; \qquad (t \in [0, 2\pi))$$

- The Dirac operator on the circle is

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

  with square $\Delta_{\mathbb{S}^1}$.
- The eigenfunctions of $D_{\mathbb{S}^1}$ in $L^2(S^1)$ are the complex exponential functions

$$e^{int} = \cos nt + i \sin nt; \qquad (n \in \mathbb{Z})$$




  and $[D_{S^1}, f] = \frac{df}{dt}$, a bounded operator on $L^2(S^1)$ for smooth $f$.

# The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- The Laplacian reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

- It seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$:

$$\left( a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2} \right)^2 = a^2\frac{\partial^2}{\partial t_1^2} + 2ab\frac{\partial^2}{\partial t_1 \partial t_2} + b^2\frac{\partial^2}{\partial t_2^2}$$

- This puzzle was solved by Dirac who considered complex *matrices*:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

then $a^2 = b^2 = -1$ and $ab + ba = 0$

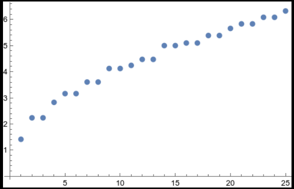
- The Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

which satisfies $(D_{\mathbb{T}^2})^2 = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}$.

- The spectrum of the Dirac operator $D_{\mathbb{T}^2}$ is

$$\left\{ \pm\sqrt{n_1^2 + n_2^2} : n_1, n_2 \in \mathbb{Z} \right\};$$




and $\|[D_{\mathbb{T}^2}, f]\| = \|f\|_{\mathrm{Lip}}$.

More generally, a Dirac operator exists on spin manifolds as a differential operator acting in $L^2(S_M)$ and square $D_M^2 = \Delta_M + \frac{1}{4}\kappa$.

# Let's go noncommutative!

- Consider two unitaries $U$ and $V$, satisfying the relation

$$UV = e^{i\theta} VU; \qquad \theta \in \mathbb{R}$$

- The **noncommutative torus** is given by

$$\mathcal{A}_\theta = \left\{ a = \sum_{mn} a_{mn} U^m V^n : (a_{mn}) \text{ rapid decay} \right\}$$

- Faithful trace $\tau : \mathcal{A}_\theta \to \mathbb{C}$ given by $a \mapsto a_{00}$ yields a Hilbert space $L^2(\mathcal{A}_\theta, \tau)$.
- Introduce derivations $\delta_1, \delta_2$ on $\mathcal{A}_\theta$ satisfying

$$\delta_1(U) = U; \qquad \delta_1(V) = 0; \qquad \delta_2(U) = 0; \qquad \delta_2(V) = V.$$

  and write

$$D_{\mathcal{A}_\theta} = \begin{pmatrix} 0 & \delta_1 + i\delta_2 \\ -\delta_1 + i\delta_2 & 0 \end{pmatrix}$$

- The triple $(\mathcal{A}_\theta, L^2(\mathcal{A}_\theta, \tau), D_{\mathcal{A}_\theta})$ describe the "smooth geometry" of the noncommutative torus [Connes, 1980].

# Spectral triples

More generally, we consider a triple $(\mathcal{A}, \mathcal{H}, D)$

- a $*$-algebra $\mathcal{A}$
- a self-adjoint operator $D$ with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A}$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ of norm 1
- Distance function on state space of $\mathcal{A}$:

$$d(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$

# Spectral truncations

joint with Connes

- More realistically, one should consider states on an **approximation** by projecting onto a frequency range around $\lambda$ with bandwidth $\Lambda$:


- The distance function still makes sense for states on such spectral truncations $P_\Lambda \mathcal{A} P_\Lambda$:

$$\boxed{d(\phi,\psi) = \sup_{\|[D,T]\|\leq 1} |\phi(T) - \psi(T)|}; \qquad T = P_\Lambda a P_\Lambda$$

- Note: $P\mathcal{A}P$ is in general not a $*$-algebra, but it is an **operator system**: there is an adjoint and a **cone** $(P\mathcal{A}P)_+$ of positive elements.

# Operator systems

Definition (Arveson 1969, Choi-Effros 1977)

*An operator system is a $*$-closed vector space $E$ of bounded operators. Unital: it contains the identity operator.*

- $E$ is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff

$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$

- in fact, $E$ is matrix ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any $n$.

Maps between operator systems $E, F$ are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all $n$.

Isomorphisms are complete order isomorphisms

# State spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on $E$ as positive linear functionals of norm 1.
- In the finite-dimensional case, the dual $E^d$ of a unital operator system is a unital operator system with

$$E_+^d = \left\{ \phi \in E^d : \phi(T) \geq 0, \forall T \in E_+ \right\}$$

and similarly for the matrix order
- It follows that we have the following useful correspondence:

$$\text{pure states on } E \longleftrightarrow \text{extreme rays in } (E^d)_+$$

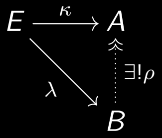
and the other way around.

# $C^*$-envelope of a unital operator system

Arveson introduced the notion of $C^*$-envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979. Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A $C^*$-extension $\kappa : E \to A$ of a unital operator system $E$ is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A $C^*$-envelope of a unital operator system is a $C^*$-extension $\kappa : E \to A$ with the following universal property:




# Shilov boundaries

There is a useful description of $C^*$-envelopes using Shilov ideals.

Definition

*Let $\kappa : E \to A$ be a $C^*$-extension of an operator system. A boundary ideal is given by a closed 2-sided ideal $I \subseteq A$ such that the quotient map $q : A \to A/I$ is completely isometric on $\kappa(E) \subseteq A$.*

*The Shilov boundary ideal is the largest of such boundary ideals.*

Proposition

*Let $\kappa : E \to A$ be a $C^*$-extension. Then there exists a Shilov boundary ideal $J$ and $C^*_{env}(E) \cong A/J$.*

As an example consider the operator system of continuous harmonic functions $C_{\mathrm{harm}}(\overline{\mathbb{D}})$ on the closed disc. Then by the maximum modulus principle the Shilov boundary is $S^1$. Accordingly, its $C^*$-envelope is $C(S^1)$.

# Example: spectral truncation of the circle

- Eigenvectors of $D_{S^1}$ are **Fourier modes** $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- **Orthogonal projection** $P_n$ onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $P_n C(S^1) P_n$ is an **operator system**
- Any such $T = P_n f P_n$ can be written as a **Toeplitz matrix**

$$P_n f P_n \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

- States are defined as unital positive linear functionals.

# Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C(S^1)_{(n)}$:

- functions on $S^1$ with a finite number of non-zero Fourier coefficients:

$$f(x) = \sum_{k=-n+1}^{n-1} a_k e^{ikx}$$

- such $f$ is positive iff $\sum_k a_k e^{ikx}$ is a positive function on $S^1 \rightsquigarrow$ cone $(C(S^1)_{(n)})_+$

Proposition

*The extreme rays in $(C(S^1)_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on $S^1$.*

# Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C(S^1)_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C(S^1)_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

*Pure states are given by vector states $T \mapsto \langle \xi, T\xi \rangle$ where (up to normalization)*

$$\xi = \begin{pmatrix} 1 & \sum_k \lambda_k & \dots & \lambda_1 \cdots \lambda_{n-1} \end{pmatrix}^t; \qquad |\lambda_k| = 1.$$

*Consequently, the pure state space $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n / S_n$.*

# Gromov–Hausdorff convergence

Recall Gromov–Hausdorff distance between two metric spaces:

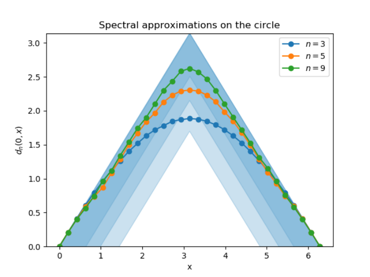
$$d_{\mathrm{GH}}(X, Y) = \inf\{d_H(f(X), g(Y)) \mid f : X \to Z, g : Y \to Z \text{ isometric}\}$$

and

$$d_H(X, Y) = \inf\{\epsilon \geq 0; X \subseteq Y_\epsilon, Y \subseteq X_\epsilon\}$$

Rieffel extends this to quantum metric spaces (essentially operator systems equipped with a Lip-norm).

# Distance function for spectral truncations of the circle




Proposition (vS21, Hekkelman 2021)

*The sequence of state spaces* $\{(\mathcal{S}(P_n C(S^1) P_n), d_n)\}$ *converges to* $(\mathcal{S}(C(S^1)), d_{S^1})$ *in Gromov–Hausdorff distance.*

And more examples include (quantum) fuzzy spheres, Fourier truncations, truncations of tori (Leimbach–vS23, RU) ...