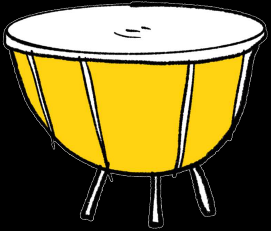

Geometry emerging from spectra

Walter van Suijlekom

Radboud Universiteit

Lorentz in October 1910

H.A. Lorentz by Jan Veth

Lorentz in October 1910

H.A. Lorentz by Jan Veth

Origins of spectral geometry:

the high overtones behave inversely proportional to the volume.

Weyl in February 1911

$$\begin{aligned} N(\Lambda) &= \#\text{wave numbers } \leq \Lambda \\ &\sim \frac{\Omega_d \mathrm{Vol}(M)}{d(2\pi)^d}\Lambda^d \end{aligned}$$

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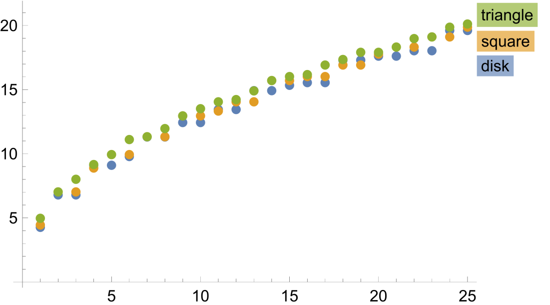
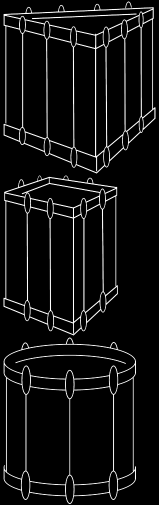
$$\sim \frac{\Omega_d \mathrm{Vol}(M)}{d(2\pi)^d}\Lambda^d$$

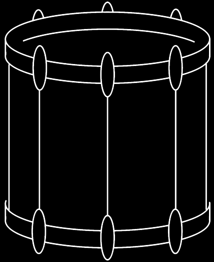
Evidence by the parabolic shapes ($\sqrt{\Lambda}$):





Mark Kac in 1966

“Can one hear the shape of a drum?”



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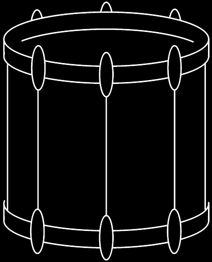
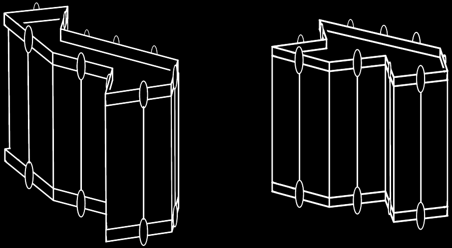


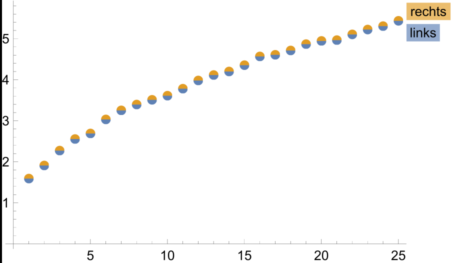
Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of M?

Isospectral drums!





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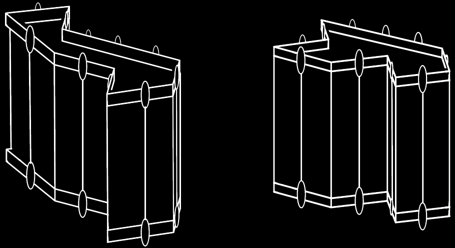

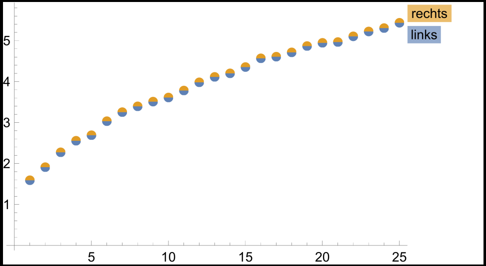

… so the answer to Kac's question is **no**
and more information is needed…

Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

- Distance $d(x, y)$ between two points is usually defined as

  > *the* **smallest** *of the arclengths (computed using the metric) of curves connecting x and y.*

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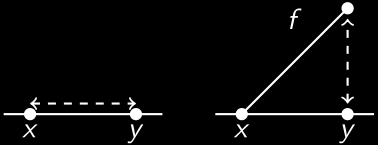

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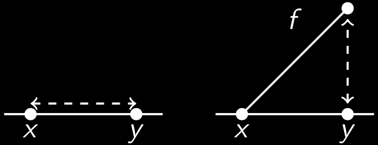
$$\boxed{d(x, y) = \sup_{\|[D_M, f]\| \leq 1} |\delta_x(f) - \delta_y(f)|}$$



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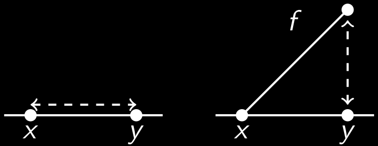
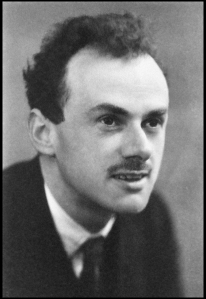


Combination $(C^\infty(M), L^2(S_M), D_M)$
allows for reconstruction of geometry

Analysis: Dirac operator

Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator is a 'square-root' of the Laplacian, so that its spectrum give the wave numbers k.
- First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold M.



The circle

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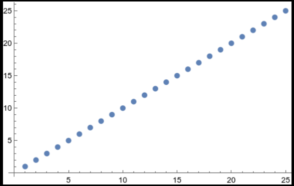
- The eigenfunctions of $D_{\mathbb{S}^1}$ in $L^2(S^1)$ are the complex exponential functions

$$e^{int} = \cos nt + i \sin nt; \qquad (n \in \mathbb{Z})$$



  and $[D_{S^1}, f] = \frac{df}{dt}$, a bounded operator on $L^2(S^1)$ for smooth f.

The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
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$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

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- It seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$:

$$\left(a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2} \right)^2 = a^2\frac{\partial^2}{\partial t_1^2} + 2ab\frac{\partial^2}{\partial t_1 \partial t_2} + b^2\frac{\partial^2}{\partial t_2^2}$$

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- This puzzle was solved by Dirac who considered complex *matrices*:
$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

then $a^2 = b^2 = -1$ and $ab + ba = 0$

- The Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

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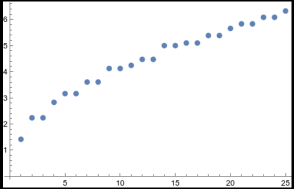
- The spectrum of the Dirac operator $D_{\mathbb{T}^2}$ is

$$\left\{ \pm\sqrt{n_1^2 + n_2^2} : n_1, n_2 \in \mathbb{Z} \right\};$$



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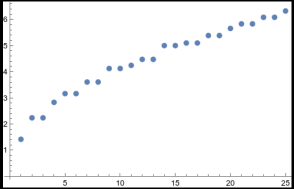

  and $\|[D_{\mathbb{T}^2}, f]\| = \|f\|_{\mathrm{Lip}}$.

More generally, a Dirac operator exists on spin manifolds as a differential operator acting in $L^2(S_M)$ and square $D_M^2 = \Delta_M + \frac{1}{4}\kappa$.

Let's go noncommutative!

- Consider two unitaries U and V, satisfying the relation

$$UV = e^{i\theta} VU; \qquad \theta \in \mathbb{R}$$

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- Introduce derivations δ_1, δ_2 on $\mathcal{A}_\theta$ satisfying

$$\delta_1(U) = U; \qquad \delta_1(V) = 0; \qquad \delta_2(U) = 0; \qquad \delta_2(V) = V.$$

and write

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- The triple $(\mathcal{A}_\theta, L^2(\mathcal{A}_\theta, \tau), D_{\mathcal{A}_\theta})$ describe the "smooth geometry" of the noncommutative torus [Connes, 1980].

Spectral triples

More generally, we consider a triple $(\mathcal{A}, \mathcal{H}, D)$

- a $*$-algebra $\mathcal{A}$
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A}$
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Generalized distance function:

- States are positive linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ of norm 1
- Distance function on state space of $\mathcal{A}$:

$$d(\phi, \psi) = \sup_{a \in \mathcal{A}} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

Spectral truncations

joint with Connes

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- Our distance function still makes sense for states on such spectral truncations $P_\Lambda \mathcal{A} P_\Lambda$:

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- Note: $P\mathcal{A}P$ is in general not a $*$-algebra, but it is an **operator system**: there is an adjoint and a **cone** $(P\mathcal{A}P)_+$ of positive elements.

Example: spectral truncation of the circle

- Eigenvectors of D_{S^1} are **Fourier modes** $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- **Orthogonal projection** P_n onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $P_n C(S^1) P_n$ is an **operator system**

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- Any such $T = P_n f P_n$ can be written as a **Toeplitz matrix**

$$P_n f P_n \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

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- States are defined as unital positive linear functionals.

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C(S^1)_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$f(x) = \sum_{k=-n+1}^{n-1} a_k e^{ikx}$$

- such f is positive iff $\sum_k a_k e^{ikx}$ is a positive function on $S^1 \rightsquigarrow$ cone $(C(S^1)_{(n)})_+$

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Proposition

The extreme rays in $(C(S^1)_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C(S^1)_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C(S^1)_{(n)} \to \mathbb{C}$$

$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

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Proposition

Pure states are given by vector states $T \mapsto \langle \xi, T\xi \rangle$ where (up to normalization)

$$\xi = \begin{pmatrix} 1 & \sum_k \lambda_k & \dots & \lambda_1 \cdots \lambda_{n-1} \end{pmatrix}^t; \qquad |\lambda_k| = 1.$$

Consequently, the pure state space $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$.

Gromov–Hausdorff convergence

Recall Gromov–Hausdorff distance between two metric spaces:

$$d_{\mathrm{GH}}(X,Y) = \inf\{d_H(f(X), g(Y)) \mid f : X \to Z, g : Y \to Z \text{ isometric}\}$$

and

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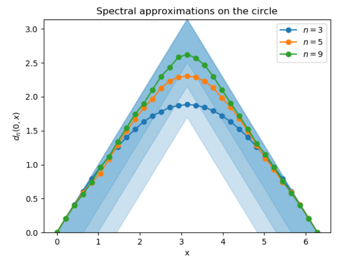
Rieffel extends this to quantum metric spaces (essentially operator systems equipped with a Lip-norm).

Distance function for spectral truncations of the circle



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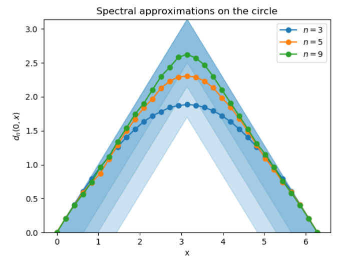

Proposition (vS21, Hekkelman 2021)

The sequence of state spaces $\{(\mathcal{S}(P_n C(S^1) P_n), d_n)\}$ *converges to* $(\mathcal{S}(C(S^1)), d_{S^1})$ *in Gromov–Hausdorff distance.*

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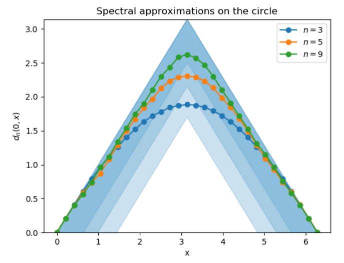

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And more examples include (quantum) fuzzy spheres, Fourier truncations, truncations of tori (Leimbach–vS23, RU) ...