Inner fluctuations of geometry: Morita equivalence and connections

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Recap: noncommutative manifolds

Basic device: a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$:

- algebra $\mathcal{A}$ of bounded operators on
- a Hilbert space $\mathcal{H}$,
- a self-adjoint operator D in $\mathcal{H}$ with compact resolvent
  and such that the commutator $[D, a]$ is bounded for all $a \in \mathcal{A}$.

Relation to K-homology and K-theory

Let A be a C^*-algebra.

Definition

A Fredholm module for A *is given by a* $*$*-representation* π *of* A *on a (graded) Hilbert space* $\mathcal{H}$ *and an (odd) operator* F *in* $\mathcal{H}$ *such that the operators*

$$[F, a], \qquad a(F^2 - 1), \qquad a(F - F^*)$$

are compact for all $a \in A$.

The K-homology groups $K^1(A)$ (and $K^0(A)$ in the graded case) are given by homotopy equivalence classes of Fredholm modules for A.

Index pairings

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ defines a class $[D] \equiv [F_D] \in K^\bullet(A)$ with $A = \overline{\mathcal{A}}$, given by

$$F_D = D(1 + D^2)^{-1/2}$$

Pairings between K-homology and K-theory become index pairings of the form :

$$\langle [D], [u] \rangle = \operatorname{index} PuP; \qquad P = \chi(D \geq 0); \qquad \text{(odd case)}$$

$$\langle [D], [p] \rangle = \operatorname{index} pD^+p; \qquad D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}; \qquad \text{(even case).}$$

Real spectral triples

$$(\mathcal{A}, \mathcal{H}, D)$$

- Extend to a *real* spectral triple:

$$J : \mathcal{H} \to \mathcal{H} \qquad \text{real structure (anti-unitary)}$$

  such that

$$J^2 = \pm 1; \qquad JD = \pm DJ$$

- *Right action* of $\mathcal{A}^{\mathrm{op}}$ on $\mathcal{H}$ via $a^{\mathrm{op}} = Ja^* J^{-1}$ and we demand

$$[a^{\mathrm{op}}, b] = 0; \qquad a, b \in \mathcal{A}$$

- D is said to satisfy the *first-order condition* if

$$[[D, a], b^{\mathrm{op}}] = 0$$

Key example: Riemannian spin geometry

Let M be an compact m-dimensional Riemannian spin manifold.

- $\mathcal{A} = C^\infty(M)$
- $\mathcal{H} = L^2(S_M)$, square integrable spinors
- $D = D_M$, Dirac operator
- $J = C$ (charge conjugation)

Gauge theory

- Action of unitaries $u \in \mathcal{U}(\mathcal{A})$ on self-adjoint operators D by

$$D \mapsto UDU^*; \qquad U = uJuJ^{-1}$$

- *Gauge group*: $\mathcal{G}(\mathcal{A}) := \{uJuJ^{-1} : u \in \mathcal{U}(\mathcal{A})\}$
- Compute *rhs*:

$$UDU^* = D + u[D, u^*] + \hat{u}[D, \hat{u}^*] + \hat{u}[u[D, u^*], \hat{u}^*]$$

with $\hat{u} = JuJ^{-1}$ and last term on rhs vanishes if D satisfies *first-order* condition

Morita equivalence

Suppose $\mathcal{A} \sim_M \mathcal{B}$.
Can we construct a *spectral triple* on $\mathcal{B}$ from $(\mathcal{A}, \mathcal{H}, D)$?

- Let $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ with $\mathcal{E}$ finitely generated projective. Define

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$$

  Then $\mathcal{B}$ acts as bounded operators on $\mathcal{H}'$.

- The self-adjoint operator $(1 \otimes_{\nabla} D)(\eta \otimes \psi) := \nabla_D(\eta)\psi + \eta \otimes D\psi$ requires a *universal connection* on $\mathcal{E}$:

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$$

  where ∇_D indicates that $a\delta(b) \in \Omega^1(\mathcal{A})$ is represented as $a[D, b] \in \Omega^1_D(\mathcal{A})$.

- Then $(\mathcal{B}, \mathcal{H}', 1 \otimes_{\nabla} D)$ is a spectral triple [Connes, 1996].

Morita equivalence

with real structure

Again, suppose $\mathcal{A} \sim_M \mathcal{B}$.

- If there is a *real structure* J on $(\mathcal{A}, \mathcal{H}, D)$, then we define

$$\mathcal{H}' := (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}) \otimes_{\mathcal{A}} \overline{\mathcal{E}}$$

  with the *conjugate* (left $\mathcal{A}$-) module $\overline{\mathcal{E}}$

- Define analogously the operator $(1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1$ on $\mathcal{H}'$, where

$$\overline{\nabla} : \overline{\mathcal{E}} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \overline{\mathcal{E}},$$

  and we also define

$$J' : \mathcal{H}' \to \mathcal{H}', \qquad \eta \otimes \psi \otimes \overline{\rho} \mapsto \rho \otimes J\psi \otimes \overline{\eta}$$

Proposition (Chamseddine–Connes–vS, 2013)

We have $(1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1 = 1 \otimes_\nabla (D \otimes_{\overline{\nabla}} 1)$ *and the tuple* $(\mathcal{B}, \mathcal{H}', (1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1; J')$ *is a real spectral triple.*

Morita self-equivalence

without real structure

- If $\mathcal{B} = \mathcal{A}$ (i.e. $\mathcal{E} = \mathcal{A}$) we have $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \simeq \mathcal{H}$.
- The operator D is perturbed to $D' \equiv D + A_{(1)}$ where

$$A_{(1)} := \nabla(1) = \sum_j a_j[D, b_j]$$

- *Gauge transformations* $D' \mapsto uD'u^*$ implemented by

$$A_{(1)} \mapsto uA_{(1)}u^* + u[D, u^*]$$

Morita self-equivalence

with real structure

- If $\mathcal{B} = \mathcal{A}$ (i.e. $\mathcal{E} = \mathcal{A}$) we have $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \overline{\mathcal{E}} \simeq \mathcal{H}$ and $J' \equiv J$.
- The operator D is perturbed to $D' \equiv D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)}$ where

$$A_{(1)} := \sum_j a_j[D, b_j], \qquad \widetilde{A}_{(1)} := \sum_j \hat{a}_j[D, \hat{b}_j] = \pm J A_{(1)} J^{-1};$$
$$A_{(2)} := \sum_j \hat{a}_j[A_{(1)}, \hat{b}_j] = \sum_{j,k} \hat{a}_j a_k[[D, b_k], \hat{b}_j]$$

  and $A_{(2)}$ vanishes if D satisfies first-order condition
- *Gauge transformations* $D' \mapsto U D' U^*$ implemented by

$$A_{(1)} \mapsto u A_{(1)} u^* + u[D, u^*]$$
$$A_{(2)} \mapsto J u J^{-1} A_{(2)} J u^* J^{-1} + J u J^{-1}[u[D, u^*], J u^* J^{-1}]$$

Semi-group of inner perturbations

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

with semi-group law inherited from product in $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.

- $\mathcal{U}(\mathcal{A})$ maps to $\mathrm{Pert}(\mathcal{A})$ by sending $u \mapsto u \otimes u^{*\mathrm{op}}$.
- $\mathrm{Pert}(\mathcal{A})$ acts on D:

$$D \mapsto \sum_j a_j D b_j$$

- For *real* spectral triples we use the map $\mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{A} \otimes \hat{\mathcal{A}})$ sending $T \mapsto T \otimes \hat{T}$ so that

$$D \mapsto \sum_{i,j} a_i \hat{a}_j D b_i \hat{b}_j$$

and this extends the action of the gauge group $\mathcal{G}(\mathcal{A})$.

Perturbation semi-group and Morita self-equivalences

Proposition (Chamseddine–Connes–vS, 2013)

- *The linear map* $\eta : \mathrm{Pert}(\mathcal{A}) \to \Omega^1(\mathcal{A})$, $\eta(a \otimes b^{\mathrm{op}}) = a\delta(b)$ *is surjective.*
- *If* $\sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathrm{Pert}(\mathcal{A})$ *then the perturbed operator*

$$\sum_j a_j D b_j = D + \sum_j a_j [D, b_j] \equiv D + A_{(1)}$$

and, for real spectral triples:

$$\sum_{i,j} a_i \hat{a}_j D b_i \hat{b}_j = D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)}$$

- *The gauge transformations are implemented via the map* $\mathcal{U}(A) \to \mathrm{Pert}(\mathcal{A})$.

Perturbation semigroup for matrix algebras

Proposition

Let $\mathcal{A}_F$ be the algebra of block diagonal matrices (fixed size). Then the perturbation semigroup of $\mathcal{A}_F$ *is*

$$\mathrm{Pert}(\mathcal{A}_F) \simeq \left\{ \sum_j A_j \otimes B_j \in \mathcal{A}_F \otimes \mathcal{A}_F \;\middle|\; \begin{array}{l} \sum_j A_j (B_j)^t = \mathbb{I} \\ \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j} \end{array} \right\}$$

The semigroup law in $\mathrm{Pert}(\mathcal{A}_F)$ *is given by the matrix product in* $\mathcal{A}_F \otimes \mathcal{A}_F$*:*

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB').$$

- The two conditions in the above definition,

$$\sum_j A_j (B_j)^t = \mathbb{I} \qquad \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j}$$

  are the *normalization* and *self-adjointness condition*, respectively.
- Let us check that the normalization condition carries over to products,

$$\left(\sum_j A_j \otimes B_j\right)\left(\sum_k A'_k \otimes B'_k\right) = \sum_{j,k} (A_j A'_k) \otimes (B_j B'_k)$$

  for which indeed

$$\sum_{j,k} A_j A'_k (B_j B'_k)^t = \sum_{j,k} A_j A'_k (B'_k)^t (B_j)^t = \mathbb{I}$$

Example: perturbation semigroup of two-point space

- Now $\mathcal{A}_F = \mathbb{C}^2$, the *algebra of diagonal* 2×2 *matrices*.
- In terms of the standard basis of such matrices

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

  we can write an arbitrary element of $\mathrm{Pert}(\mathbb{C}^2)$ as

$$z_1 e_{11} \otimes e_{11} + z_2 e_{11} \otimes e_{22} + z_3 e_{22} \otimes e_{11} + z_4 e_{22} \otimes e_{22}$$

- Matrix multiplying e_{11} and e_{22} yields for the normalization condition:

$$z_1 = 1 = z_4.$$

- The self-adjointness condition reads

$$z_2 = \overline{z_3}$$

  leaving only one free complex parameter so that $\mathrm{Pert}(\mathbb{C}^2) \simeq \mathbb{C}$.

Example: perturbations for two-point space

- An element $\phi \in \mathbb{C} \cong \mathrm{Pert}(\mathbb{C}^2)$ acts on D_F as:

$$D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \overline{c}\overline{\phi} \\ c\phi & 0 \end{pmatrix}$$

- The group of unitary matrices is now $U(1) \times U(1)$ and an element (λ, μ) therein acts as

$$\phi \mapsto \overline{\lambda}\mu \cdot \phi$$

- The field ϕ is a so-called abelian Higgs field.

Example: perturbations for noncommutative two-point space

- Consider *noncommutative two-point space* described by $\mathbb{C} \oplus M_2(\mathbb{C})$
- It turns out that

$$\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C})) \simeq M_2(\mathbb{C}) \times \mathrm{Pert}(M_2(\mathbb{C}))$$

- Only $M_2(\mathbb{C}) \subset \mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ acts non-trivially on D_F:

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \overline{c\phi_1} & \overline{c\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$

- Physicists call ϕ_1 and ϕ_2 the *Higgs field*.
- The group of unitary block diagonal matrices is now $U(1) \times U(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Example: perturbations for $M_N(\mathbb{C})$

Consider $(M_N(\mathbb{C}), \mathbb{C}^N, D)$ with D a hermitian matrix, not proportional to the identity.

- Inner perturbation are of the form $D \mapsto \sum_j a_j D b_j = D + \sum_j a_j [D, b_j]$
- Note that
$$e_{kl} = \frac{1}{\lambda_j - \lambda_l} e_{kj}[D, e_{jl}] \in \Omega^1_D(\mathcal{A})$$
for some j (so that $\lambda_j \neq \lambda_l$).
- Thus $\Omega^1_D(\mathcal{A}) \cong M_N(\mathbb{C})$, in particular, we may take $-D \in \Omega^1_D(\mathcal{A})$ and perturb $D \mapsto 0$.
- Since $\Omega^1_0(\mathcal{A}) = 0$ all subsequent inner perturbations vanish..

In general, Morita equivalence of C^*-algebras does not lift to equivalence between spectral triples

Examples: Pert($\mathcal{A}$) for function algebras

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

- If $\mathcal{A} = C^\infty(M)$ then

$$\mathrm{Pert}(C^\infty(M)) \simeq \left\{ f \in C^\infty(M \times M) \;\middle|\; \begin{array}{ll} f(x,x) = 1 & \forall x \in M \\ f(x,y) = \overline{f(y,x)} & \forall x, y \in M \end{array} \right\}$$

- Action of $\mathrm{Pert}(C^\infty(M))$ on $D = D_M = i\gamma^\mu \nabla_\mu$ is given by

$$\sum_j a_j D_M b_j = D_M + i\gamma^\mu \nabla_{\frac{\partial}{\partial y^\mu}} f(x,y)|_{x=y} =: D_M + i\gamma^\mu A_\mu$$

with $A_\mu \in C^\infty(M, \mathfrak{u}(1))$

Non-abelian Yang–Mills theory

On a 4-dimensional background:

- $\mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C})$
- $\mathcal{H} = L^2(S) \otimes M_n(\mathbb{C})$
- $D = D_M \otimes 1$
- $J = C \otimes (.)^*$.

Proposition (Chamseddine-Connes, 1996)

- *The perturbations* $A_{(1)} + \widetilde{A}_{(1)}$ *with* $A_{(1)} = \gamma^\mu A_\mu$ *describes an* $\mathfrak{su}(n)$*-gauge field on* M*.*
- *Gauge group* $\mathcal{G}(\mathcal{A}) \simeq C^\infty(M, SU(n))$

The spectral Standard Model

Describe geometry of $M \times F_{SM}$ by $(C^\infty(M) \otimes A_F, L^2(S_M) \otimes \mathcal{H}_F, D_{M\times F})$ [CCM 2007]

- Coordinates in $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (with gauge group $U(1)_Y \times SU(2)_L \times SU(3)$).
- Dirac operator $D_{M\times F} = D_M \otimes 1 + \gamma_M \otimes D_F$ where

$$D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

is a 96×96-dimensional hermitian matrix where 96 is:

$$3 \times 2 \times (\underline{2} \otimes \underline{1} + \underline{1} \otimes \underline{1} + \underline{1} \otimes \underline{1} + \underline{2} \otimes \underline{3} + \underline{1} \otimes \underline{3} + \underline{1} \otimes \underline{3})$$

The Dirac operator on F_{SM}

$$D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

- The operator S is given by

$$S_l := \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ Y_\nu^* & 0 & 0 & 0 \\ 0 & Y_e^* & 0 & 0 \end{pmatrix}, \qquad S_q \otimes 1_3 = \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ Y_u^* & 0 & 0 & 0 \\ 0 & Y_d^* & 0 & 0 \end{pmatrix} \otimes 1_3,$$

  where Y_ν, Y_e, Y_u and Y_d are 3×3 mass matrices acting on the three generations.
- The symmetric operator T only acts on the right-handed (anti)neutrinos, $T\nu_R = Y_R \overline{\nu_R}$ for a 3×3 symmetric Majorana mass matrix Y_R, and $Tf = 0$ for all other fermions $f \neq \nu_R$.

Inner perturbations

- Inner perturbations of D_M give a matrix

$$A_\mu = \begin{pmatrix} B_\mu & 0 & 0 & 0 \\ 0 & W_\mu^3 & W_\mu^+ & 0 \\ 0 & W_\mu^- & -W_\mu^3 & 0 \\ 0 & 0 & 0 & (G_\mu^a) \end{pmatrix}$$

  corresponding to hypercharge, weak and strong interaction.
- Inner perturbations of D_F give

$$\begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \rightsquigarrow \begin{pmatrix} Y_\nu \phi_1 & -Y_e \overline{\phi}_2 \\ Y_\nu \phi_2 & Y_e \overline{\phi}_1 \end{pmatrix}$$

  corresponding to SM-Higgs field. Similarly for Y_u, Y_d.
- Beyond the Standard Model: Grand algebra unification, Pati–Salam, etc.