Geometric spaces at finite resolution

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Spectral geometry: origins

H.A. Lorentz door Jan Veth

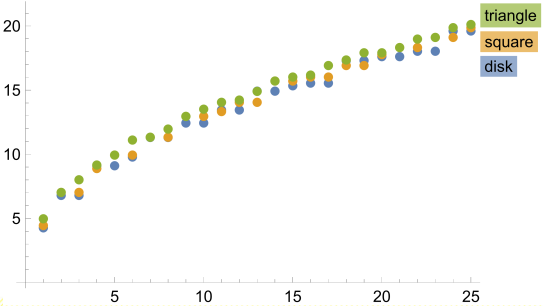
"Hierbei entseht das mathematische Problem, zu beweisen, dass die Anzahl der genügend hohen Obertöne zwischen n und $n+dn$ unabhängig von der Gestalt der Hülle und nur ihrem Volumen proportional ist."

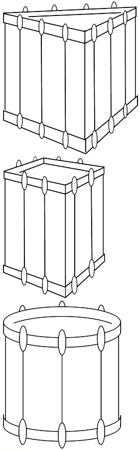
"Here arises the mathematical problem of proving that the number of sufficiently high harmonics between n and $n+dn$ is independent of the shape of the envelope and proportional only to its volume."

Weyl's Law

$$N(\Lambda) = \#\text{wave numbers} \ \le \Lambda$$
$$\sim \frac{\Omega_d \mathrm{Vol}(M)}{d(2\pi)^d} \Lambda^d$$

Evidence by the parabolic shapes ($\sqrt{\Lambda}$):





A spectral approach to geometry

"Can one hear the shape of a drum?" (Kac, 1966)

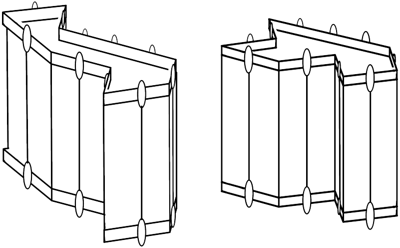
Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

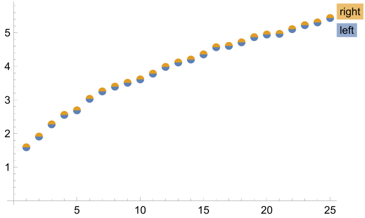
$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of M?

Similarly, for a Riemannian spin manifold and Dirac operator D_M (so that $D_M^2 = \Delta_M + \frac{1}{4}\kappa$)

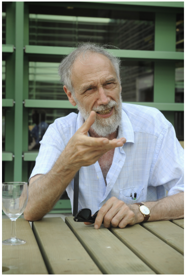
Isospectral drums





so answer to Kac's question is no

Noncommutative geometry



If combined with the C^*-algebra $C(M)$, then the answer to Kac's question is affirmative.

Connes' reconstruction theorem [2008]:

$$(C(M), L^2(S_M), D_M) \longleftrightarrow (M, g)$$

The spectral approach to geometry

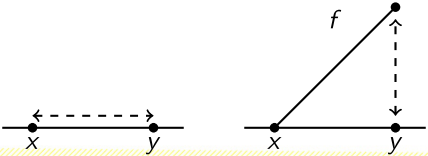
Given cpt Riemannian spin manifold (M, g) with spinor bundle S_M on M.

- the C^*-algebra $C(M)$
- the self-adjoint Dirac operator D_M
- both acting on Hilbert space $L^2(S_M)$

$\rightsquigarrow$ spectral triple: $(C(M), L^2(S_M), D_M)$

Reconstruction of distance function [Connes 1994]:

$$d(x,y) = \sup_{f \in C(M)} \{|\mathrm{ev}_x(f) - \mathrm{ev}_y(f)| : \|[D_M, f]\| \leq 1\}$$



Spectral triples

More generally, we consider a triple $(A, \mathcal{H}, D)$

- a C^*-algebra A
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : A \to \mathbb{C}$ of norm 1
- Pure states are extreme points of state space
- Distance function on state space of A:

$$d(\phi, \psi) = \sup_{a \in A} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

Spectral data

- The mathematical reformulation of geometry in terms of spectral data (global analysis) requires the knowledge of the full Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.
- We aim for the underlying mathematical formalism for doing (noncommutative) geometry with only part of the spectrum and/or with finite resolution.

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged)

Towards operator systems..

(I) Given $(A, \mathcal{H}, D)$ we project onto part of the spectrum of D:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

(II) Another approach would be to consider metric spaces up to a finite resolution :

- Consider integral operators associated to the tolerance relation R_ϵ given by $d(x, y) < \epsilon$

So first, some background on operator systems.

Operator systems

Definition (Arveson 1969, Choi-Effros 1977)
An operator system is a $*$-closed vector space E of bounded operators.
Unital: it contains the identity operator.

- E is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff
$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$
- in fact, E is matrix ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any n.

Maps between operator systems E, F are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

Isomorphisms are complete order isomorphisms

State spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on E as positive linear functionals of norm 1.
- In the finite-dimensional case, the dual E^d of a unital operator system is a unital operator system with

$$E^d_+ = \{\phi \in E^d : \phi(T) \geq 0, \forall T \in E_+\}$$

  and similarly for the matrix order
- It follows that we have the following useful correspondence:

  pure states on $E \longleftrightarrow$ extreme rays in $(E^d)_+$

  and the other way around.

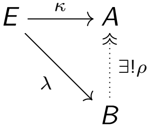
In the infinite-dimensional/non-unital case, this is more subtle (more later)...

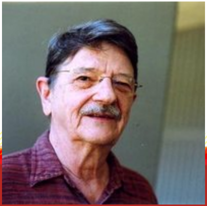
C^*-envelope of a unital operator system

Arveson introduced the notion of C^*-envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979.
Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A C^*-extension $\kappa : E \to A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of a unital operator system is a C^*-extension $\kappa : E \to A$ with the following universal property:





Shilov boundaries

There is a useful description of C^*-envelopes using Shilov ideals.

Definition

Let $\kappa : E \to A$ be a C^*-extension of an operator system. A boundary ideal is given by a closed 2-sided ideal $I \subseteq A$ such that the quotient map $q : A \to A/I$ is completely isometric on $\kappa(E) \subseteq A$.

The Shilov boundary ideal is the largest of such boundary ideals.

Proposition

Let $\kappa : E \to A$ be a C^*-extension. Then there exists a Shilov boundary ideal J and $C^*_{env}(E) \cong A/J$.

As an example consider the operator system of continuous harmonic functions $C_{\text{harm}}(\overline{\mathbb{D}})$ on the closed disc. Then by the maximum modulus principle the Shilov boundary is S^1. Accordingly, its C^*-envelope is $C(S^1)$.

Spectral truncation of the circle [Connes-vS, 2020]

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\text{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C^*_{\text{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.
- The C^*-envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z})$

Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ $(\lambda \in S^1)$.*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. *The extreme rays in* $C(S^1)^{(n)}_+$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n / S_n$.

Curiosities on Toeplitz matrices

Theorem (Carathéodory)

Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ iff $T = V \Delta V^*$ with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \qquad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \dots, d_n \geq 0$ and $\lambda_1, \dots, \lambda_n \in S^1$.

Spaces at finite resolution [Connes-vS, 2021]

Consider now a path metric measure space X with a measure of full support, and the following tolerance relation:

$$\boxed{\mathcal{R}_\epsilon := \{(x,y) \in X \times X : d(x,y) < \epsilon\}}$$

It gives rise to the operator system $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$.

Proposition

If X is a complete and locally compact path metric measure space X with a measure of full support. Then $C^*_{env}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$. The pure states of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is ϵ-connected.

Non-unital operator systems

Consider a matrix-ordered operator space $(E, \|\cdot\|)$.

- The noncommutative (nc) state space is defined for any n as

$$\mathcal{S}_n(E) := \{\phi \in M_n(E)^*, \|\phi\| = 1, \phi \geq 0\}$$

not always convex
nor weakly $*$-compact

- The nc quasi-state space is defined for any n as

$$\widetilde{\mathcal{S}}_n(E) := \{\phi \in M_n(E)^*, \|\phi\| \leq 1, \phi \geq 0\}$$

convex
and weakly $*$-compact

- The modified numerical radius $\nu_E : M_n(E) \to \mathbb{C}$ is defined as

$$\nu_E(x) = \sup\left\{ \left|\phi\left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix}\right)\right| : \phi \in \widetilde{\mathcal{S}}_{2n}(E) \right\}.$$

Definition (Werner)
A non-unital operator system is given by a matrix-ordered operator space for which $\nu_E(\cdot) = \|\cdot\|$.

Approximate order units

We now consider a particular class of non-unital operator systems.

Definition (Ng 1969)

Let E be a matrix-ordered $*$-vector space. An approximate order unit for E is a net $\{e_\lambda\}_{\lambda \in \Lambda}$ with the following properties

1. $e_\lambda \in E_+$ *for all* $\lambda \in \Lambda$ *and* $e_\lambda \leq e_\mu$ *whenever* $\lambda \leq \mu$;
2. *for each* $x \in E_h$ *there exists a positive real number* t *and* $\lambda \in \Lambda$ *such that*

$$-te_\lambda \leq x \leq te_\lambda.$$

In fact, if the approximate order unit is norm defining in the sense that

$$\|x\| = \inf\left\{t : \begin{pmatrix} te_\lambda^n & x \\ x^* & te_\lambda^n \end{pmatrix} \in M_{2n}(E)_+ \text{ for some } \lambda \in \Lambda\right\}$$

then E is a non-unital operator system [Karn 2005, Han 2010].

Assuming the existence of a norm-defining approximate order unit in E we may show familiar C^*-results such as

1. the nc state space $S_n(E)$ is convex

and if $E \subseteq A$ with a norm-defining approximate order unit for A contained in E we have that

2. any (pure) state on E can be extended to a (pure) state on A.
3. we have an isometrical order isomorphism

$$\boxed{A^*/E^\perp \to E^*}$$

This also applies if we replace E and A by dense subspaces $\mathcal{E}$ and $\mathcal{A}$.

Operator systems associated to tolerance relations

- Key motivating example: a metric space (X, d) with the relation

$$\boxed{\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}}$$

- If (X, μ) is a measure space and $\mathcal{R}_\epsilon \subseteq X \times X$ an open subset we obtain the operator system $E(\mathcal{R}_\epsilon)$ as the closure of integral operators with support in $\mathcal{R}_\epsilon$. Note that $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$

Finite partial partitions of a metric measure space

A finite partial ϵ-partition of X is a finite collection $P = \{U\}$ of disjoint measurable sets $U \subseteq X$ such that $\mathrm{diam}(U) < \epsilon$; directed by refinement.

- The corresponding finite-dimensional algebra $\mathcal{A}_P$ with unit e_P is

$$\mathcal{A}_P = \left\{ \sum_{U,V \in P} a_{UV} |1_U\rangle\langle 1_V| : a_{UV} \in \mathbb{C} \right\} \cong \mathcal{K}(l^2(P))$$

- A tolerance relation $\mathcal{R}_\epsilon^P$ on the finite set P is given by

$$\mathcal{R}_\epsilon^P = \{U \times V \mid \quad U, V \in P, \quad U \times V \subseteq \mathcal{R}_\epsilon\} \subseteq P \times P$$

  and yields the (unital) operator system $E(\mathcal{R}_\epsilon^P) \subseteq \mathcal{A}_P$.
- If $P \leq P'$ then $E(\mathcal{R}_\epsilon^P) \subseteq E(\mathcal{R}_\epsilon^{P'})$ and also $\mathcal{A}_P \subseteq \mathcal{A}_{P'}$.
- Approximate order unit $\{e_P\}_P$ of $\varinjlim A_P$ is contained in $\varinjlim E(\mathcal{R}_\epsilon^P)$

Spaces at finite resolution [Connes-vS, 2021]

Proposition

Let X be a path metric measure space with a measure of full support.

1. $\mathcal{E}(\mathcal{R}_\epsilon) := \varinjlim E(\mathcal{R}_\epsilon^P)$ *is a dense subspace of* $E(\mathcal{R}_\epsilon)$
2. $\mathcal{A}_\epsilon := \varinjlim \mathcal{A}_P$ *is a dense* $*$*-subalgebra of the* C^**-algebra* $\mathcal{K}(L^2(X))$*;*
3. *there exists a norm-defining approximate order unit for* $\mathcal{A}_\epsilon$ *which is contained in* $\mathcal{E}(\mathcal{R}_\epsilon)$*.*

Proposition

Let X be a complete, locally compact path metric measure space with a measure of full support. Then $C^*_{env}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$*.*
The pure states of $E(\mathcal{R}_\epsilon)$ *are given by vector states* $|\psi\rangle\langle\psi|$ *where the essential support of* $\psi \in L^2(X)$ *is* ϵ*-connected.*