Operator systems and noncommutative geometry

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The spectral approach to geometry

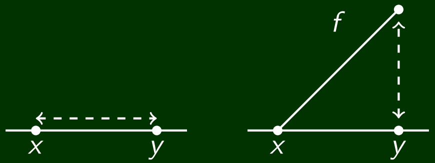
Given cpt Riemannian spin manifold (M, g) with spinor bundle S_M on M.

- the C^*-algebra $C(M)$
- the self-adjoint Dirac operator D_M
- both acting on Hilbert space $L^2(S_M)$

$\rightsquigarrow$ spectral triple: $(C(M), L^2(S_M), D_M)$

Reconstruction of distance function [Connes 1994]:

$$d(x,y) = \sup_{f \in C(M)} \{ |\mathrm{ev}_x(f) - \mathrm{ev}_y(f)| : \|[D_M, f]\| \leq 1 \}$$



Spectral triples

More generally, we consider a triple $(A, \mathcal{H}, D)$

- a C^*-algebra A
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : A \to \mathbb{C}$ of norm 1
- Pure states are extreme points of state space
- Distance function on state space of A:

$$d(\phi, \psi) = \sup_{a \in A} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$

Spectral data: $(\mathcal{A}, \mathcal{H}, D)$

- The mathematical reformulation of geometry in terms of spectral data (global analysis) requires the knowledge of the full Dirac operator.
- From a physical or computational standpoint this is not very realistic: detectors have limited energy ranges and resolution.
- We aim for the underlying mathematical formalism for doing (noncommutative) geometry with only part of the spectrum. [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged, ...)

Thus, we project onto part of the spectrum of D:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

Operator systems

Definition (Arveson 1969, Choi-Effros 1977)

An operator system is a $*$-closed vector space E of bounded operators. Unital: it contains the identity operator.

- E is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff

$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$

- in fact, E is matrix ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any n.

Maps between operator systems E, F are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

Isomorphisms are complete order isomorphisms

C^*-envelope of a unital operator system

Arveson introduced the notion of C^*-envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979. Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A C^*-extension $\kappa : E \to A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of a unital operator system is a C^*-extension $\kappa : E \to A$ with the following universal property:

$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & \searrow^{\lambda} & \uparrow \exists! \rho \\ & & B \end{array}$$

C^*-envelope via boundary representations (à la Arveson)

Dritschel–McCullough 2005, Arveson 2008, Davidson–Kennedy 2015

View E as a concrete operator system in the C^*-algebra $C^*(E)$ it generates.

- A ucp map $\phi : E \to B(\mathcal{H})$ has the *unique extension property* if is has a unique ucp extension to $C^*(E)$ which is a $*$-representation
- If, in addition, the $*$-representation is irreducible, it is called a *boundary representation*

Theorem

A unital operator system E is completely normed by its boundary representations, that is, for every $x \in M_n(E)$ there is a boundary representation π of E such that $\|x\| = \|\pi^{(n)}(x)\|$.

Corollary

The direct sum of all boundary representations of E yield a completely isometric map $\imath : E \to B(\mathcal{H})$ such that $(C^*(\imath(E)), \imath)$ is the C^*-envelope of E.

Example: spectral truncation of the circle [Connes-vS, 2020]

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\text{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C^*_{\text{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C(S^1)_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\dots, 0, a_{-n+1}, a_{-n+2}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-2}, a_{n-1}, 0, \dots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.
- The C^*-envelope of $C(S^1)_{(n)}$ is given by $C^*(\mathbb{Z})$

Proposition

1. *The extreme rays in $(C(S^1)_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C(S^1)_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C(S^1)_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C(S^1)_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. *The extreme rays in* $C(S^1)_+^{(n)}$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$.

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Towards K-theory for operator systems

A key invariant of C^*-algebras is K-theory. Is there an analogue for operator systems?

- Need notion of projection or invertible selfadjoint elements (*cf.* Araiza–Russell)
- It should capture the spectral localizer of Loring, Schulz-Baldes, and others
- It should be invariant under Morita equivalence [EKT]

Definition

A hermitian form x *in a unital operator system* E *is a selfadjoint element* $x \in M_n(E)$ *which is* non-degenerate *in the sense that there exists* $g > 0$ *such that for all pure and maximal ucp maps* $\phi : E \to B(\mathcal{H})$ *we have*

$$|\phi^{(n)}(x)| \geq g \cdot \mathrm{id}_{\mathcal{H}}^{\oplus n}$$

In other words, x *should have a gap* g *in each boundary representation*

We will write $H(E,n)$ for all hermitian forms in $M_n(E)$.

Proposition

An element $x \in M_n(E)$ is non-degenerate if and only if $\imath_E^{(n)}(x)$ is an invertible element in the C^*-envelope $C^*_{\mathrm{env}}(E)$.

This is a consequence of the realization of the C^*-envelope in [Davidson–Kennedy]

Examples:

1. Hermitian forms (à la Witt) on a fgp right module pA^n over a C^*-algebra A: described by invertible elements $x = h + (1-p) \in M_n(A)$ with $h \in pM_n(A)p$.
2. Projections p in operator systems à la Araiza–Russell are projections in the C^*-envelope: $x = e - 2p$ is a hermitian form. Similarly, for ϵ-projections.
3. Spectral compressions of projections in C^*-algebra: $x = PYP$ with $Y = 1 - 2p$ provided $\|[P,p]\|$ sufficiently small.

The invariants and K-theory

$$\boxed{\mathcal{V}(E,n) = \pi_0(H(E,n)).}$$

Example:

$$\mathcal{V}(\mathbb{C},n) \cong \{-n, -n+2, \ldots, n\}$$

and with the map $\imath_{nm}([x] = x \oplus e_{m-n}$ we have

$$\begin{array}{ccc} \mathcal{V}(\mathbb{C},n) & \xrightarrow{\ \imath_{nm}\ } & \mathcal{V}(\mathbb{C},m) \\ & {\scriptstyle \rho_n} \searrow \quad \swarrow {\scriptstyle \rho_m} & \\ & \mathbb{Z} & \end{array}$$

In general, we consider

$$\mathcal{V}(E) = \varinjlim \mathcal{V}(E,n)$$

and $K_0(E)$ is the corresponding Grothendieck group (with identity $[e]$ and addition $'\oplus'$)

Properties of K_0

- For C^*-algebras we obtain usual K-theory via the map $[x] \mapsto [p = \frac{1}{2}(1 - x|x|^{-1})]$.
- Stability: we define a map $\imath_n : M_n(E) \to M_n(M_2(E))$ by

$$\imath_n(x) = \left(\begin{array}{cc|cc|c|cc} x_{11} & 0 & x_{12} & 0 & \dots & x_{1n} & 0 \\ 0 & e & 0 & 0 & & 0 & 0 \\ \hline x_{21} & 0 & x_{22} & 0 & \dots & \multicolumn{2}{c}{\vdots} \\ 0 & 0 & 0 & e & & & \\ \hline \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c}{\vdots} \\ \hline x_{n1} & 0 & \multicolumn{2}{c|}{\dots} & \dots & x_{nn} & 0 \\ 0 & e & & & & 0 & e \end{array}\right)$$

so that $\imath_n(x) \sim x$ (Whitehead). This allows to show $K_0(E) \cong K_0(M_2(E))$.

Non-unital operator systems

The unitization [Werner, 2002] of a non-unital operator system E is given by the $*$-vector space $E^+ = E \oplus \mathbb{C}$ with matrix order structure:

$$(x, A) \geq 0 \text{ iff } A \geq 0 \text{ and } \phi(A_\epsilon^{-1/2} x A_\epsilon^{-1/2}) \geq -1$$

for all $\epsilon > 0$ and noncommutative states $\phi \in \mathcal{S}_n(E)$, and where $A_\epsilon = \epsilon \mathbb{I}_n + A$.

$$\widetilde{\mathcal{V}}(E, n) := \pi_0 \left(\{ (x, A) \in H(E^+, n) : A \sim_n \mathbb{I}_n \} \right)$$

In the unital case, the isomorphism $E^+ \cong E \oplus \mathbb{C}$ given by $(x, A) \mapsto (x + Ae, A)$ yields that in this case

$$\widetilde{\mathcal{V}}(E, n) \cong \mathcal{V}(E, n).$$

Theorem

For a unital operator system E we have $K_0(\mathcal{K} \otimes E) \cong K_0(E)$.

Stability

Theorem

For a unital operator system E we have $K_0(\mathcal{K} \otimes E) \cong K_0(E)$.

Proof.

1. Realize stabilization by maps $\kappa_{NM} : M_N(E) \to M_M(E), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0_{M-N} \end{pmatrix}$
2. Commuting diagram:

$$\begin{array}{ccc} \widetilde{V}(E,n) & \xrightarrow{\kappa_{1N}} & \widetilde{\mathcal{V}}(M_N(E),n) \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{V}(E,n) & \xrightarrow[\cong]{\imath_n} & \mathcal{V}(M_N(E),n) \end{array}$$

3. The map $\kappa_{1\infty} : K_0(E) \to K_0(\mathcal{K} \otimes E)$ is an isomorphism:
   - injective: homotopy in $H((\mathcal{K} \otimes E)^+, n)$ compressed to homotopy in $H((M_N(E)^+, n)$.
   - surjective: approximation by finite-rank operators in norm is still hermitian form.

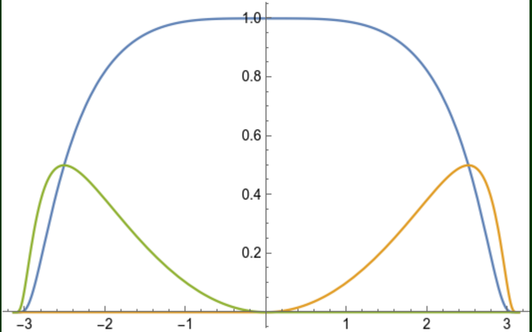
Example: spectral localizer on the 2-torus

Loring considered (in his thesis!) analogues of the Powers–Rieffel projections:

$$p = \begin{pmatrix} f & g + hU \\ g + hU^* & 1 - f \end{pmatrix} \in M_2(C^\infty(\mathbb{T}^2))$$

with U a unitary in the second variable, and f, g, h real-valued smooth functions in the first variable, satisfying

$$gh = 0, \qquad g^2 + h^2 = f - f^2.$$



Spectral truncations on $\mathbb{T}^2$

- We now consider spectral truncations $P = P_\rho$ onto $\ell^2\{\vec{n} \in \mathbb{Z}^2 : \|\vec{n}\| \leq \rho\} \subseteq l^2(\mathbb{Z}^2)$.
- We obtain a compression PYP of the hermitian form $Y = 1 - 2p$ on $\mathbb{T}^2$ corresponding to p:

$$PYP = \begin{pmatrix} P - 2PfP & -2PgP - 2PhUP \\ -2PgP - 2PhU^*P & -P + 2PfP \end{pmatrix} \in M_2(PC^\infty(\mathbb{T}^2)P)$$

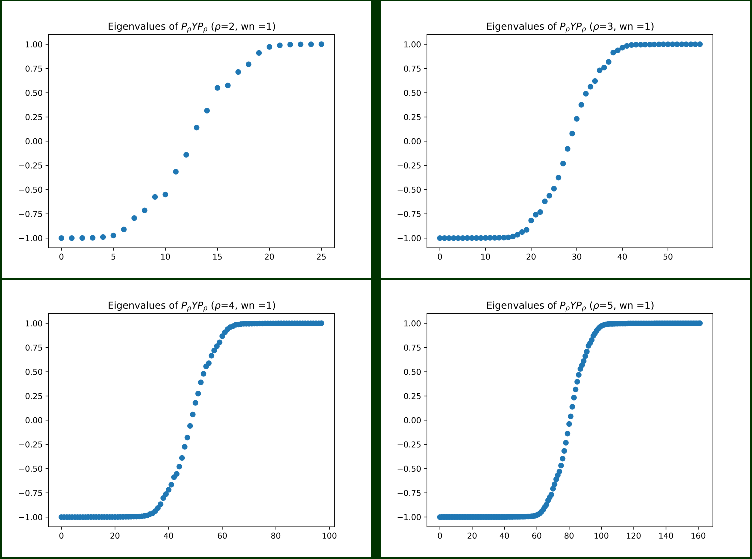
  For suitable P these are hermitian forms $\rightsquigarrow [PYP] \in K_0(PC(\mathbb{T}^2)P)$.
- The *spectral localizer* of Loring and Schulz-Baldes is given by the following matrix:

$$L_{\kappa,\rho} = \begin{pmatrix} -PYP & \kappa PD^+P \\ \kappa PD^-P & PYP \end{pmatrix}$$

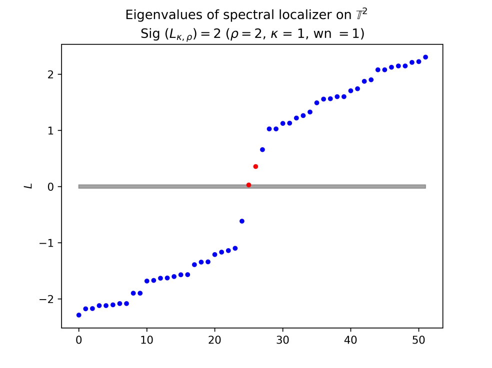
  In general they show that for suitable κ and ρ the index pairing can be computed as the signature of this matrix:

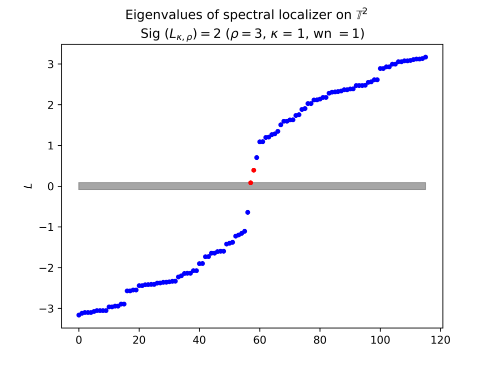
$$\boxed{\text{Index } pD^+p = \frac{1}{2}\text{Sig } L_{\kappa,\rho}}$$

Simulations: eigenvalues of PYP for $U(t_2) = e^{it_2}$



Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{it_2}$





Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{2it_2}$