Spectral geometry and quantum theory I

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Lorentz in October 1910

H.A. Lorentz by Jan Veth

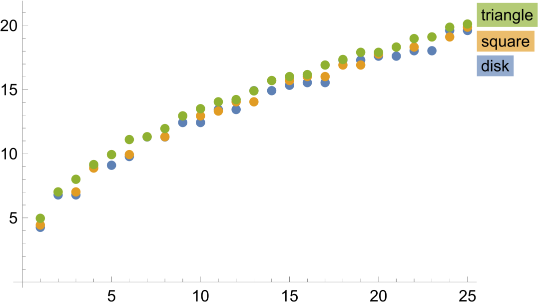
Origins of spectral geometry:

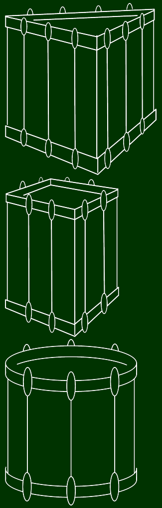
"Hierbei entseht das mathematische Problem, zu beweisen, dass die Anzahl der genügend hohen Obertöne zwischen n *und* $n + dn$ *unabhängig von der Gestalt der Hülle und nur ihrem Volumen proportional ist."*

Weyl in February 1911

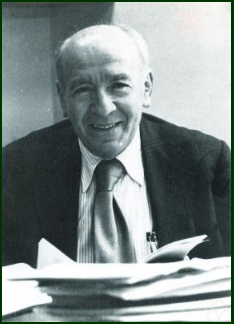
$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$
$$\sim \frac{\Omega_d \mathrm{Vol}(M)}{d(2\pi)^d}\Lambda^d$$

Evidence by the parabolic shapes ($\sqrt{\Lambda}$):

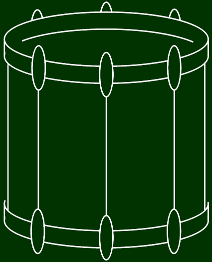




Mark Kac in 1966



“Can one hear the shape of a drum?”

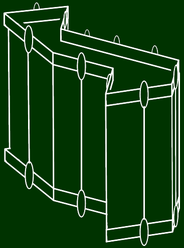


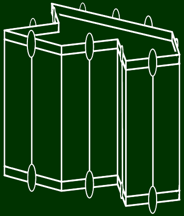
Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

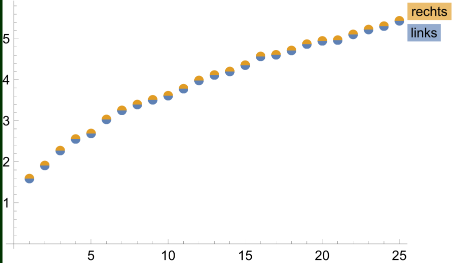
$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of M?

Isospectral drums!







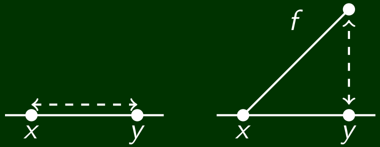
… so the answer to Kac’s question is **no** and more information is needed…

Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

- Distance $d(x, y)$ between two points is usually defined as
  the **smallest** *of the arclengths (computed using the metric) of curves connecting x and y.*
- But it can also be defined as
  the **largest** *of differences $|f(x) - f(y)|$ for functions f with gradient $|\nabla f| \leq 1$.*

$$\boxed{d(x, y) = \sup_{\|[D_M, f]\| \leq 1} |\delta_x(f) - \delta_y(f)|}$$



Combination $(C^\infty(M), L^2(S_M), D_M)$ allows for reconstruction of geometry

Spectral triples

More generally, we consider a triple $(\mathcal{A}, \mathcal{H}, D)$

- a $*$-algebra $\mathcal{A}$
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A}$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ of norm 1
- Distance function on state space of $\mathcal{A}$:

$$d(\phi, \psi) = \sup_{a \in \mathcal{A}} \left\{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \right\}$$

Example: **commutative** two-point space

$$F = {}_1\bullet \qquad\qquad {}_2\bullet$$

- Then the algebra of smooth functions

$$C^\infty(F) := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

- A finite Dirac operator is given by

$$D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}; \qquad (c \in \mathbb{C})$$

- The distance formula then becomes

$$d(1,2) = \frac{1}{|c|}$$

Unitary (CKM) invariant: $\mathrm{Spec}_N(M)$

[Connes 2008]

In general, all structure of a Riemannian spin manifold can be captured by a pair of

- The spectrum of the Dirac operator
- The relative position of two commutative von Neumann algebras: a discrete and a continuous one.

Pair of maximal abelian von Neumann algebras in finite dimension

- Consider a pair of maximal abelian subalgebas $M, N \subseteq L(\mathcal{H})$ with $\dim(\mathcal{H}) = n$.
- Corresponding to each idempotent $e_x \in M, f_\lambda \in N$ there are unit vectors $\xi_x, \eta_\lambda \in \mathcal{H}$ (up to a phase).
- They yield unitary matrices $C_{x\lambda} := \langle \xi_x, \eta_\lambda \rangle$
- Each row $C_{x\bullet}$ is an element in the unit sphere $\mathbb{S}_1(N)$ of N, since

  $$\sum_\lambda |C_{x\lambda}|^2 = 1$$

  and we are interested in the set of equivalence classes of these n rows in $\mathbb{P}(N) := \mathbb{S}_1(N)/U(1)$.
- Up to the gauge action of $\mathcal{U}(N)$ on the projective space, this defines the invariant $\mathrm{Spec}_N(M)$,

Partition of unity

- A convenient encoding of these equivalence classes is given by the hermitian matrices of rank 1 defined by:

  $$(\rho_x)_{\lambda\mu} = \overline{C_{x\lambda}} C_{x\mu}$$

  which satisfy:

  $$\rho_x^2 = \rho_x; \qquad \rho_x \rho_y = 0 \quad (x \neq y); \qquad \sum_x (\rho_x)_{\lambda\mu} = \delta_{\lambda\mu} \qquad (*)$$

In fact, we have

Proposition (Connes, 2008)

Let N be a maximal abelian von Neumann subalgebra of $L(\mathcal{H})$. The relative position of abelian von Neumann algebras of constant multiplicity m in $L(\mathcal{H})$ is classified by subsets $Spec_N(M)$ in the rank m hermitian matrices ρ (matrix indices in $Spec(N)$) with n/m elements such that $(*)$ holds, up to the adjoint action of $\mathcal{U}(N)$.

Example: Cabibbo-invariant for two-point space

- maximal abelian subalgebra $A = M$ in $L(\mathbb{C}^2)$ and let $N = \mathbb{C}^2$ be diagonal, corresponding to $D = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$.
- We write $C = \begin{pmatrix} |\psi_1\rangle^t \\ |\psi_2\rangle^t \end{pmatrix}$ where

$$|\psi_1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}; \qquad |\psi_2\rangle = \begin{pmatrix} -\overline{\beta} \\ \overline{\alpha} \end{pmatrix}; \qquad |\alpha|^2 + |\beta|^2 = 1.$$

- All phases of α, β can be gauged away, leaving them to be $c, s \in \mathbb{R}$ with $c^2 + s^2 = 1$ and two commuting idempotents:

$$\rho_1 = |\psi_1\rangle\langle\psi_1| = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}; \qquad \rho_2 = |\psi_2\rangle\langle\psi_2| = \begin{pmatrix} s^2 & -cs \\ -cs & c^2 \end{pmatrix}.$$

- For the distance function we write $a = a_1\rho_1 + a_2\rho_2 \in M$ and compute: $d(1,2) = (|\lambda| \sin(2\theta))^{-1}$ in terms of $\theta \in [0, \Pi)$ (the 'Cabibbo angle').

Example: **noncommutative** two-point space

The two-point space can be given a noncommutative structure by considering the algebra A_F of 3×3 block diagonal matrices of the following form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}$$

A finite Dirac operator for this example is given by a hermitian 3×3 matrix, for example

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

… a good old noncommutative space!

- Consider two unitaries U and V, satisfying the relation

$$UV = e^{i\theta} VU; \qquad \theta \in \mathbb{R}$$

- The **noncommutative torus** [Connes 1980, Rieffel 1981] is given by

$$\mathcal{A}_\theta = \left\{ a = \sum_{mn} a_{mn} U^m V^n : (a_{mn}) \text{ rapid decay } \right\}$$

- Faithful trace $\tau : \mathcal{A}_\theta \to \mathbb{C}$ given by $a \mapsto a_{00}$ yields a Hilbert space $L^2(\mathcal{A}_\theta, \tau)$.
- Introduce derivations δ_1, δ_2 on $\mathcal{A}_\theta$ satisfying

$$\delta_1(U) = U; \qquad \delta_1(V) = 0; \qquad \delta_2(U) = 0; \qquad \delta_2(V) = V.$$

  and write

$$D_{\mathcal{A}_\theta} = \begin{pmatrix} 0 & \delta_1 + i\delta_2 \\ -\delta_1 + i\delta_2 & 0 \end{pmatrix}$$

- The triple $(\mathcal{A}_\theta, L^2(\mathcal{A}_\theta, \tau) \otimes \mathbb{C}^2, D_{\mathcal{A}_\theta})$ describes the "smooth geometry" of the noncommutative torus [Connes, 1980].

Spectral data: $(\mathcal{A}, \mathcal{H}, D)$

- The mathematical reformulation of geometry in terms of spectral data (global analysis) requires the knowledge of the full Dirac operator.
- From a physical or computational standpoint this is not very realistic: detectors have limited energy ranges and resolution.
- We develop the underlying mathematical formalism for doing (noncommutative) geometry with only part of the spectrum. [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged, ...)

Thus, we project onto part of the spectrum of D:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

Operator systems

Definition (Arveson 1969, Choi-Effros 1977)

An operator system is a $*$-closed vector space E of bounded operators. Unital: it contains the identity operator.

- E is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff

$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$

- in fact, E is matrix ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any n.

Maps between operator systems E, F are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

Isomorphisms are complete order isomorphisms

C^*-envelope of a unital operator system

Arveson 1969, Hamana 1979, …

A C^*-extension $\kappa : E \to A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of a unital operator system is a C^*-extension $\kappa : E \to A$ with the following universal property:

$$\begin{tikzcd}
E \arrow[r, "\kappa"] \arrow[dr, "\lambda"'] & A \\
& B \arrow[u, dotted, two heads, "\exists!\rho"']
\end{tikzcd}$$

Example: operator system $C_{\text{harm}}(\overline{\mathbb{D}})$ of continuous harmonic functions with C^*-envelope $C(S^1)$.

Example: spectral truncation of the circle [Connes-vS, 2020]

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

- States are defined as unital positive linear functionals.

We have: $C^*_{\mathrm{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C(S^1)_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.
- The C^*-envelope of $C(S^1)_{(n)}$ is given by $C^*(\mathbb{Z}) \cong C(S^1)$

Proposition

1. *The extreme rays in* $(C(S^1)_{(n)})_+$ *are given by the elements* $a = (a_k)$ *for which the Laurent series* $\sum_k a_k z^k$ *has all its zeroes on* S^1*.*
2. *The pure states of* $C(S^1)_{(n)}$ *are given by* $a \mapsto \sum_k a_k \lambda^k$ $(\lambda \in S^1)$*.*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C(S^1)_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C(S^1)_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. *The extreme rays in* $C(S^1)^{(n)}_+$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$.

Curiosities on Toeplitz matrices

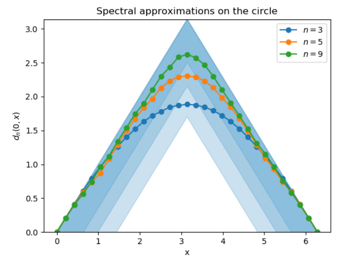
Theorem (Carathéodory)

Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ iff $T = V \Delta V^*$ with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \qquad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \dots, d_n \geq 0$ and $\lambda_1, \dots, \lambda_n \in S^1$.

Distance function for spectral truncations of the circle



Proposition (vS21, Hekkelman 2021)

The sequence of state spaces $\{(\mathcal{S}(P_n C(S^1) P_n), d_n)\}$ *converges to* $(\mathcal{S}(C(S^1)), d_{S^1})$ *in Gromov–Hausdorff distance.*

And more examples include (quantum) fuzzy spheres, Fourier truncations, truncations of tori, Peter–Weyl truncations (Gaudillot-Estrada, Leimbach, RU) ...

Spectral geometry and quantum theory II

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Chapter One.

Fermions

Spectral triples: one-particle (fermionic) physical systems

- A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ allows to write the Dirac equation for $\psi \in \mathcal{H}$:

$$\boxed{D\psi = 0}$$

- This describes the propagation of a single fermion (described by the 'wave function' ψ): one-particle quantum physics.
- We keep this first quantization as a mystery, and will consider second quantization as a functor (free on Edward Nelson).
- Let us start with some basic theory on second quantization [Araki 1987, Carey 1987, GVF 2001]

Clifford algebra: from $\mathcal{H} \to \mathbb{C}\mathrm{l}(\mathcal{H}_{\mathbb{R}})$

- Given a Hilbert space $\mathcal{H}$, consider the real Euclidean vector space underlying it: $V = \mathcal{H}_{\mathbb{R}}$.
- In finite dimensions, we may define $\mathbb{C}\mathrm{l}(V) = \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$ as the complexified Clifford algebra, generated by $c(v)$ for any $v \in V$ with relations:

$$c(v)c(w) + c(w)c(v) = 2\langle v, w\rangle$$

- In the infinite-dimensional case, we consider the algebraic Clifford algebra $\mathbb{C}\mathrm{l}^{\mathrm{alg}}(V) = \cup \mathbb{C}\mathrm{l}(V_f)$ with the union over all finite-dimensional subspaces $V_f \subseteq V$.
- This $*$-algebra is equipped with a trace τ such that $\tau(1) = 1$, we define $\mathbb{C}\mathrm{l}(V)$ as the C^*-closure in $B(H_\tau)$ for the Hilbert space completion H_τ of $\mathbb{C}\mathrm{l}^{\mathrm{alg}}(V)$. [Plymen–Robinson, 1994]

C^*-dynamical system and KMS-condition

- The self-adjoint operator D in $\mathcal{H}$ is used as the generator of a one-parameter group of automorphisms of the Clifford algebra:

$$\sigma_t^D(c(v)) = c(e^{itD}v); \qquad (v \in V_f)$$

- In short the KMS condition at inverse temperature β means that one has the formal equality

$$\varphi(a\sigma_t(b))|_{t=i\beta} = \varphi(ba) \qquad a, b \in \mathcal{C}.$$

Proposition

For any $\beta > 0$ there exists a unique KMS$_\beta$ state φ_β on the C^*-dynamical system $(\mathbb{C}\mathrm{l}(\mathcal{H}_\mathbb{R}), \sigma_t^D)$.

Proposition

If the operator $\exp(-\beta|D|)$ is of trace class, the state φ_β is of type I and the associated irreducible representation is given by the fermionic second quantization associated to the complex structure $I := i\,\mathrm{sign}\,D$ on $\mathcal{H}_\mathbb{R}$.

Fermionic second quantization

- Equip $\mathcal{H}_{\mathbb{R}}$ with complex structure, *e.g.* $I = i \operatorname{sign} D \rightsquigarrow$ Dirac sea e^{ItD}
- $\mathbb{C}\mathrm{l}(\mathcal{H}_{\mathbb{R}})$ acts on the Fock space $\bigwedge \mathcal{H}_I$ via

$$\gamma_I(v) = a_I^*(v) + a_I(v); \qquad (v \in \mathcal{H}_{\mathbb{R}}).$$

Proposition (Chamseddine–Connes–vS, 2018)

(*i*) *The one-parameter group* σ_t^D *is implemented in the (physical) Fock representation by the one-parameter unitary group* $\bigwedge \exp(it|D|)$*:*

$$\gamma_I(\sigma_t^D(A)) = \bigwedge(e^{it|D|})\gamma_I(A)\bigwedge(e^{-it|D|}) \qquad A \in \mathbb{C}\mathrm{l}(\mathcal{H}_{\mathbb{R}}).$$

(*ii*) *If* $\exp(-\beta|D|)$ *is of trace class the state* φ_β *is of type I and is given by*

$$\varphi_\beta(A) = \mathcal{N}^{-1}\operatorname{Tr}\left(\bigwedge \exp(-\beta|D|)\gamma_I(A)\right) \qquad A \in \mathbb{C}\mathrm{l}(\mathcal{H}_{\mathbb{R}})$$

Gibbs states and entropy

- We thus have a density matrix

$$\rho_\beta = \mathcal{N}^{-1} \cdot \bigwedge(e^{-\beta|D|})$$

- Note that this is the Gibbs state for a Fermi gas on the (noncommutative) space that is described by $(\mathcal{A}, \mathcal{H}, D)$.

Theorem (Chamseddine–Connes–vS, 2018)

The (von Neumann) entropy,

$$S(\rho_\beta) = -\operatorname{Tr} \rho_\beta \log \rho_\beta,$$

*of the above Gibbs state ρ_β is given by a spectral action $\mathrm{Tr}_{\mathcal{H}}\, h(\beta D)$ for the function $h(x) = \mathcal{E}(e^{-x})$ where $\mathcal{E}(y)$ is the entropy of a partition of the unit interval in two intervals with size of ratio y (*i.e. *of size $1/(1+y)$ and $y/(1+y)$).*

Analysis of the function h

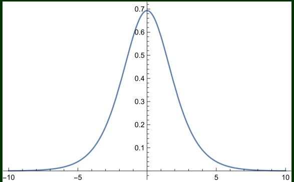
- $\mathcal{E}(y)$ is the entropy of a partition of the unit interval in two intervals with size of ratio y:

$$\mathcal{E}(y) = -\operatorname{Tr} \rho_y \log \rho_y; \qquad \rho_y = \begin{pmatrix} \frac{1}{1+y} & 0 \\ 0 & \frac{y}{1+y} \end{pmatrix}.$$

- We have $\mathcal{E}(y) = \log(y+1) - \frac{y \log y}{y+1}$

$$h(x) = \mathcal{E}(e^{-x}) = \frac{x}{1+e^x} + \log(1+e^{-x})$$

and this is applied to the spectrum of βD.



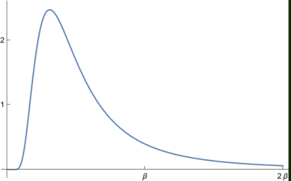
Example: entropy of two-point space

$$F = \ {}_1\bullet \qquad {}_2\bullet$$

- Distance $r := d(1,2) = 1/|c|$ in terms of $D_F = \begin{pmatrix} 0 & \bar{c} \\ c & 0 \end{pmatrix}$.
- For $r \to 0$ we have $S(\rho_\beta) = 0$;
- For $r \to \infty$ we have maximum entropy $S(\rho_\beta) = 2\log 2$;

Entropic force $F(r) = \beta^{-1}\partial_r S(\rho_\beta)$?

$$F(r) = \frac{\beta^3/2r^3}{\cosh^2(\beta/2r)}$$



Laplace transform and heat expansion

Proposition (Chamseddine–Connes–vS, 2018)

The function h is a Laplace transform:

$$h(x) = \int_0^\infty g(t) e^{-tx^2}$$

with

$$g(t) = \frac{-1}{8\sqrt{\pi}t^{5/2}} \sum_{n\in\mathbb{Z}} (-1)^n n^2 q^{n^2}; \qquad q = e^{-1/4t}.$$

This allows us to use heat asymptotics of $e^{-t\beta^2 D^2}$ to determine $\operatorname{Tr} h(\beta D)$.

Asymptotic expansion of entropy

If $\operatorname{Tr} e^{-tD^2} \sim \sum_k t^k b_k$ then

$$S(\rho_\beta) = \operatorname{Tr} h(\beta D) \sim \sum_k \beta^{2k} \gamma(k) b_k \qquad \gamma(k) = \frac{1 - 2^{-2k}}{k} \pi^{-k} \xi(2k)$$

in terms of the Riemann ξ-function :

$$\xi(s) := \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$$

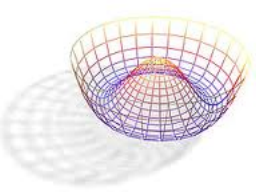
$\gamma(-1)$	$\gamma(-1/2)$	$\gamma(0)$	$\gamma(1/2)$	$\gamma(1)$	$\gamma(3/2)$
$\frac{9\zeta(3)}{2}$	$\frac{\pi^{3/2}}{3}$	$\log 2$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{8}$	$\frac{7\zeta(3)}{8\pi^{5/2}}$

Example: entropy of the electroweak theory

Use $D^2_{M\times F} = D^2_M + D^2_F$ with noncommutative two-point space $F = \mathbb{C} \oplus \mathbb{H}$ to compute (in 4d)

$$S(\rho_\beta) = \mathrm{Tr}\, h(\beta D_{M\times F}) \sim c_4 \beta^{-4} \mathrm{Vol}(M) + c_2 \beta^{-2} \int R\sqrt{g}$$
$$- c_0 \int C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \int c'_0 \,\mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} - c'_2 \beta^{-2} |\phi|^2 + c'_0 |\phi|^4 + \cdots$$

- (Higher-derivative) gravity
- The Yang–Mills term $F_{\mu\nu}F^{\mu\nu}$ for hypercharge and W-boson
- The Higgs potential $-\mu^2|\phi|^2 + \lambda|\phi|^4$



A dictionary and outlook

one-particle	second-quantized
$\mathcal{A}$	$\sigma_t^D \mapsto \sigma_t^{D_A}$
$\mathcal{H}$	$\mathbb{C}\mathrm{l}(\mathcal{H}_\mathbb{R})$
D	$\{\sigma_t^D\}_t$ arising from e^{itD}
spectral action	entropy of KMS

- Geometric significance of this entropy
- Extension to type II?
- Quantization of inner perturbations

Chapter Two.

Bosons

Real spectral triples

$$(\mathcal{A}, \mathcal{H}, D)$$

- Extend to a *real* spectral triple:

  $$J : \mathcal{H} \to \mathcal{H} \qquad \text{real structure (anti-unitary)}$$

  such that

  $$J^2 = \pm 1; \qquad JD = \pm DJ$$

- *Right action* of $\mathcal{A}^{\mathrm{op}}$ on $\mathcal{H}$ via $a^{\mathrm{op}} = Ja^* J^{-1}$ and we demand

  $$[a^{\mathrm{op}}, b] = 0; \qquad a, b \in \mathcal{A}$$

- D is said to satisfy the *first-order condition* if

  $$[[D, a], b^{\mathrm{op}}] = 0$$

Key example: AC geometries

Let M be an compact m-dimensional Riemannian spin manifold.

- $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$
- $\mathcal{H} = L^2(S_M) \otimes \mathcal{H}_F$
- $D = D_M \otimes 1 + \gamma_M \otimes D_F$, Dirac operator
- $J = C_M \otimes J_F$ (charge conjugation)

Gauge theory

- Action of unitaries $u \in \mathcal{U}(\mathcal{A})$ on self-adjoint operators D by

$$D \mapsto UDU^*; \qquad U = uJuJ^{-1}$$

- *Gauge group*: $\mathcal{G}(\mathcal{A}) := \{uJuJ^{-1} : u \in \mathcal{U}(\mathcal{A})\}$
- Compute *rhs*:

$$UDU^* = D + u[D, u^*] + \hat{u}[D, \hat{u}^*] + \hat{u}[u[D, u^*], \hat{u}^*]$$

with $\hat{u} = JuJ^{-1}$ and last term on rhs vanishes if D satisfies *first-order* condition

Morita equivalence

Suppose $\mathcal{A} \sim_M \mathcal{B}$.
Can we construct a *spectral triple* on $\mathcal{B}$ from $(\mathcal{A}, \mathcal{H}, D)$?

- Let $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ with $\mathcal{E}$ finitely generated projective. Define

  $$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$$

  Then $\mathcal{B}$ acts as bounded operators on $\mathcal{H}'$.
- The self-adjoint operator $(1 \otimes_{\nabla} D)(\eta \otimes \psi) := \nabla_D(\eta)\psi + \eta \otimes D\psi$ requires a *universal connection* on $\mathcal{E}$:

  $$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$$

  where ∇_D indicates that $a\delta(b) \in \Omega^1(\mathcal{A})$ is represented as $a[D, b] \in \Omega^1_D(\mathcal{A})$.
- Then $(\mathcal{B}, \mathcal{H}', 1 \otimes_{\nabla} D)$ is a spectral triple [Connes, 1996].

Morita equivalence

with real structure

Again, suppose $\mathcal{A} \sim_M \mathcal{B}$.

- If there is a *real structure* J on $(\mathcal{A}, \mathcal{H}, D)$, then we define

  $$\mathcal{H}' := (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}) \otimes_{\mathcal{A}} \overline{\mathcal{E}}$$

  with the *conjugate* (left $\mathcal{A}$-) module $\overline{\mathcal{E}}$
- Define analogously the operator $(1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1$ on $\mathcal{H}'$, where

  $$\overline{\nabla} : \overline{\mathcal{E}} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \overline{\mathcal{E}},$$

  and we also define

  $$J' : \mathcal{H}' \to \mathcal{H}', \qquad \eta \otimes \psi \otimes \overline{\rho} \mapsto \rho \otimes J\psi \otimes \overline{\eta}$$

Proposition (Chamseddine–Connes–vS, 2013)

We have $(1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1 = 1 \otimes_\nabla (D \otimes_{\overline{\nabla}} 1)$ *and the tuple* $(\mathcal{B}, \mathcal{H}', (1 \otimes_\nabla D) \otimes_{\overline{\nabla}} 1; J')$ *is a real spectral triple.*

Morita self-equivalence

without real structure

- If $\mathcal{B} = \mathcal{A}$ (i.e. $\mathcal{E} = \mathcal{A}$) we have $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \simeq \mathcal{H}$.
- The operator D is perturbed to $D' \equiv D + A_{(1)}$ where

$$A_{(1)} := \nabla_D(1) = \sum_j a_j [D, b_j]$$

- *Gauge transformations* $D' \mapsto uD'u^*$ implemented by

$$A_{(1)} \mapsto uA_{(1)}u^* + u[D, u^*]$$

Morita self-equivalence

with real structure

- If $\mathcal{B} = \mathcal{A}$ (i.e. $\mathcal{E} = \mathcal{A}$) we have $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \overline{\mathcal{E}} \simeq \mathcal{H}$ and $J' \equiv J$.
- The operator D is perturbed to $D' \equiv D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)}$ where

$$A_{(1)} := \sum_j a_j[D, b_j], \qquad \widetilde{A}_{(1)} := \sum_j \hat{a}_j[D, \hat{b}_j] = \pm J A_{(1)} J^{-1};$$

$$A_{(2)} := \sum_j \hat{a}_j[A_{(1)}, \hat{b}_j] = \sum_{j,k} \hat{a}_j a_k[[D, b_k], \hat{b}_j]$$

  and $A_{(2)}$ vanishes if D satisfies first-order condition
- *Gauge transformations* $D' \mapsto U D' U^*$ implemented by

$$A_{(1)} \mapsto u A_{(1)} u^* + u[D, u^*]$$
$$A_{(2)} \mapsto J u J^{-1} A_{(2)} J u^* J^{-1} + J u J^{-1}[u[D, u^*], J u^* J^{-1}]$$

- Spectral action functional: $\operatorname{Tr} f(D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)})$.

The perturbative expansion of the spectral action

Theorem (van Nuland–vS, 2021)

For a finitely-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and f in a suitable function class, there is the following absolutely convergent series expansion in $A = \pi_D(\omega)$:

$$\mathrm{Tr}(f(D+A) - f(D)) = \sum_{k=1}^{\infty} \left(\int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(\omega) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right)$$

Here the higher-dimensional *Chern–Simons forms* are given by

$$\mathrm{cs}_{2k-1}(\omega) := \int_0^1 \omega \cdot F_t^{k-1} dt; \qquad F_t = t\delta\omega + t^2\omega^2$$

The functionals ψ_{2k-1} and ϕ_{2k} turn out to define odd and even cyclic cocycles.

Gauge invariance

- For the Yang–Mills terms, it turns out that gauge invariance is a consequence of the fact that ϕ_{2k} are Hochschild cocycles:

$$\int_{\phi_{2k}} uF^k u^* = \int_{\phi_{2k}} F^k.$$

- Since the spectral action is a spectral invariant, it is in particular invariant under gauge transformations.
- This combines: also the Chern–Simons terms are gauge invariant:

$$\sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(uAu^* + u\delta u^*) = \sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A)$$

Theorem (van Nuland–vS, 2021)

Let f be in a suitable function class. Then the sequence $(\widetilde{\psi}_{2k+1})$ defines an odd entire cocycle and the pairing of $(\widetilde{\psi}_{2k+1})$ with $K_1(\mathcal{A})$ is trivial

Conclusions

- Noncommutative geometry: metric aspect
- Operator systems: from (spectral) truncations and tolerance relations
- New invariants: propagation number, K-theory
- Second quantization and entropy: spectral action functional
- Morita self-equivalence and gauge theory