

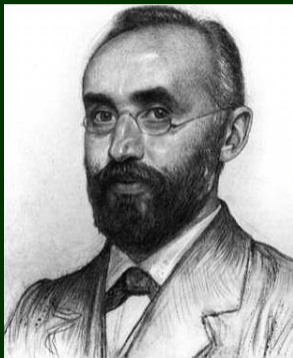
Operator systems and noncommutative geometry

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Lorentz in October 1910



H.A. Lorentz by Jan Veth

Origins of spectral geometry:

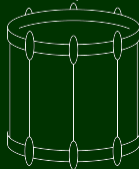
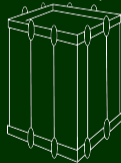
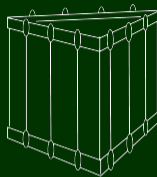
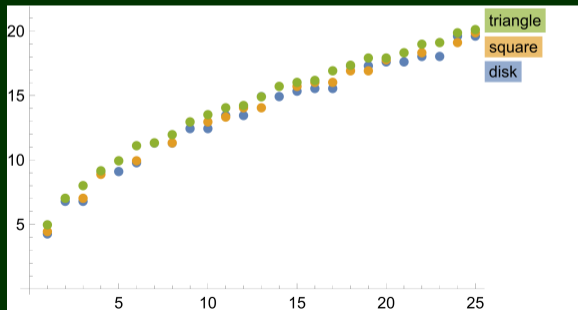
“Hierbei entseht das mathematische Problem, zu beweisen, dass die Anzahl der genügend hohen Obertöne zwischen n und $n + dn$ unabhängig von der Gestalt der Hülle und nur ihrem Volumen proportional ist.”

Weyl in February 1911

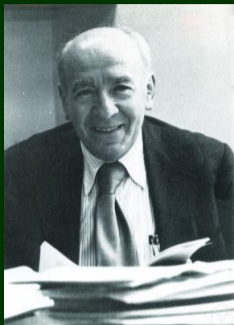
$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$

$$\sim \frac{\Omega_d \text{Vol}(M)}{d(2\pi)^d} \Lambda^d$$

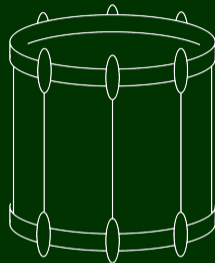
Evidence by the parabolic shapes ($\sqrt{\Lambda}$):



Mark Kac in 1966



“Can one hear the shape of a drum?”

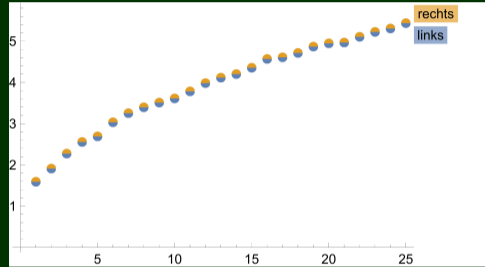
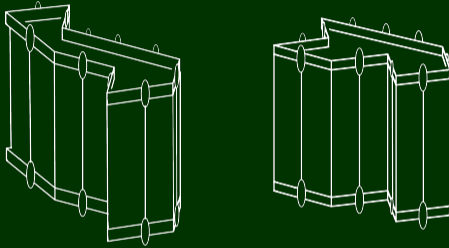


Or, more precisely, given a Riemannian manifold M , does the spectrum of wave numbers k in the Helmholtz equation

$$\Delta_M u = k^2 u$$

determine the geometry of M ?

Isospectral drums!

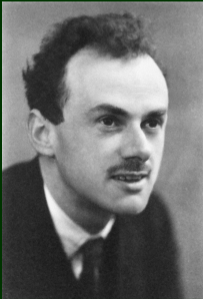


... so the answer to Kac's question is **no**
and more information is needed...

Analysis: Dirac operator

Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- ▶ The Dirac operator is a 'square-root' of the Laplacian, so that its spectrum give the wave numbers k .
- ▶ First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold M .



The circle

- ▶ The Laplacian on the circle \mathbb{S}^1 is given by

$$\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}; \quad (t \in [0, 2\pi))$$

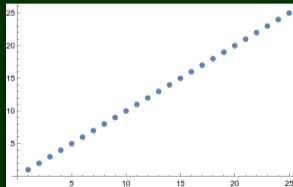
- ▶ The Dirac operator on the circle is

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

with square $\Delta_{\mathbb{S}^1}$.

- ▶ The eigenfunctions of $D_{\mathbb{S}^1}$ in $L^2(S^1)$ are the complex exponential functions

$$e^{int} = \cos nt + i \sin nt; \quad (n \in \mathbb{Z})$$



and $[D_{\mathbb{S}^1}, f] = \frac{df}{dt}$, a bounded operator on $L^2(S^1)$ for smooth f .

The 2-dimensional torus

- ▶ Consider the two-dimensional torus \mathbb{T}^2 parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- ▶ The Laplacian reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

- ▶ Dirac suggested to consider operators of the form $D_{\mathbb{T}^2} = a \frac{\partial}{\partial t_1} + b \frac{\partial}{\partial t_2}$ with complex *matrices* as coefficients:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

then $a^2 = b^2 = -1$ and $ab + ba = 0$

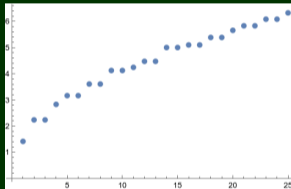
- ▶ The Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

which satisfies $(D_{\mathbb{T}^2})^2 = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}$.

- ▶ The spectrum of the Dirac operator $D_{\mathbb{T}^2}$ is

$$\left\{ \pm \sqrt{n_1^2 + n_2^2} : n_1, n_2 \in \mathbb{Z} \right\};$$



and $\|[D_{\mathbb{T}^2}, f]\| = \|f\|_{\text{Lip}}$.

More generally, a Dirac operator exists on spin manifolds as a differential operator acting in $L^2(S_M)$ and square $D_M^2 = \Delta_M + \frac{1}{4}\kappa$ for which furthermore

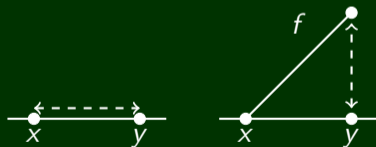
$$\|[D_M, f]\| = \|f\|_{\text{Lip}}$$

Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

- ▶ Distance $d(x, y)$ between two points is usually defined as *the **smallest** of the arclengths (computed using the metric) of curves connecting x and y .*
- ▶ But it can also be defined as *the **largest** of differences $|f(x) - f(y)|$ for functions f with gradient $|\nabla f| \leq 1$.*

$$d(x, y) = \sup_{\| [D_M, f] \| \leq 1} |\delta_x(f) - \delta_y(f)|$$



Combination $(C^\infty(M), L^2(S_M), D_M)$
allows for reconstruction of geometry

Spectral triples

More generally, we consider a triple $(\mathcal{A}, \mathcal{H}, D)$

- ▶ a unital $*$ -algebra \mathcal{A}
- ▶ a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A}$
- ▶ both acting (boundedly, resp. unboundedly) on Hilbert space \mathcal{H}

Generalized distance function:

- ▶ States are positive linear functionals $\phi : \mathcal{A} \rightarrow \mathbb{C}$ of norm 1
- ▶ Distance function on state space $S(\mathcal{A})$ of \mathcal{A} :

$$d_D(\phi, \psi) = \sup_{a \in \mathcal{A}} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

These notions continue to make sense when we replace \mathcal{A} by any self-adjoint vector space \mathcal{E} of bounded operators on \mathcal{H} that contains the unit, a so-called *operator system*.

Spectral data: $(\mathcal{A}, \mathcal{H}, D)$

- ▶ The mathematical reformulation of geometry in terms of spectral data (global analysis) requires the knowledge of the full Dirac operator.
- ▶ From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.
- ▶ We aim for the underlying mathematical formalism for doing (noncommutative) geometry with only part of the spectrum and/or with finite resolution.

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged)

Operator systems

- (I) Given (A, \mathcal{H}, D) we project onto part of the spectrum of D :
- ▶ $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
 - ▶ $D \mapsto PDP$, still a self-adjoint operator
 - ▶ $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

- (II) Another approach would be to consider metric spaces up to a finite resolution :
- ▶ Consider integral operators associated to the tolerance relation R_ϵ given by $d(x, y) < \epsilon$

C^* -envelope of a unital operator system

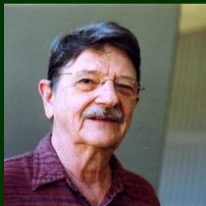
[Arveson, 1969]

Hamana: existence and uniqueness in 1979; realized á la Arveson as direct sum of all boundary representations [Dritschel–McCullough 2005, Arveson 2008, Davidson–Kennedy 2015]

A C^* -extension $\kappa : E \rightarrow A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^* -envelope of a unital operator system is a C^* -extension $\kappa : E \rightarrow A$ with the following universal property:

$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & \searrow \lambda & \uparrow \exists! \rho \\ & & B \end{array}$$



Example: operator system $C_{\text{harm}}(\overline{\mathbb{D}})$ of continuous harmonic functions with C^* -envelope $C(S^1)$.

How far is an operator system from a C^* -algebra?

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E .

Definition

The propagation number $\text{prop}(E)$ of E is defined as the smallest integer n such that $(\iota_E(E))^{\circ n} \subseteq C_{\text{env}}^(E)$ is a C^* -algebra.*

Returning to the harmonic functions in the disk we have $\text{prop}(C_{\text{harm}}(\overline{\mathbb{D}})) = 1$.

Proposition (Connes-vS, 2020; Pawłowska, 2024)

The propagation number is invariant under complete order isomorphisms, as well as under stable=Morita equivalence [EKT, 2019]:

$$\text{prop}(E) = \text{prop}(E \otimes_{\min} \mathcal{K})$$

More generally [Koot, 2021], we have

$$\text{prop}(E \otimes_{\min} F) = \max\{\text{prop}(E), \text{prop}(F)\}$$

Example: spectral truncation of the circle [Connes-vS, 2020]

- ▶ Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- ▶ Orthogonal projection $P = P_n$ onto $\text{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_n\}$
- ▶ The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- ▶ Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim (t_{k-l})_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

- ▶ States are defined as unital positive linear functionals.

We have: $C_{\text{env}}^*(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\text{prop}(C(S^1)^{(n)}) = 2$ (for any n).

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- ▶ functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\dots, 0, a_{-n+1}, a_{-n+2}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-2}, a_{n-1}, 0, \dots)$$

- ▶ an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1 .
- ▶ The C^* -envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z}) \cong C(S^1)$; propagation number ∞

Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1 .*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \rightarrow \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. The extreme rays in $C(S^1)^{(n)}_+$ are $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ for any $\lambda \in S^1$.
2. The pure state space $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n / S_n$.

Q: Does the sequence $\mathcal{S}(C(S^1)^{(n)})$ converge (in some sense) to $\mathcal{S}(C(S^1))$? (cf. Leimbach)

Operator systems, groupoids and bonds (aka a positivity domain)

Definition (Connes-vS, 2021)

A bond is a triple (G, ν, Ω) consisting of a locally compact groupoid G , a Haar system $\nu = \{\nu_x\}$ and an open symmetric subset $\Omega \subseteq G$ containing the units $G^{(0)}$.

Proposition

Let (Ω, G, ν) be a bond. The closure of the subspace $C_c(\Omega) \subseteq C_c(G)$ in the C^* -algebra $C^*(G)$ is a (possibly non-unital) operator system.

Example

1. Consider Ω in a l.c. Lie group $G \rightsquigarrow$ Fourier truncations (à la Rieffel) in $C^*(G)$
2. Consider $\Omega_n = (-n, n) \subset \mathbb{Z} \rightsquigarrow$ Fejér–Riesz system $C^*(\mathbb{Z})_{(n)} \cong (C(S^1)^{(n)})^d$.
3. Consider $\Omega_n = (-n, n) \subseteq C_m$ (so modulo m). The operator system consists of banded $m \times m$ circulant matrices of band width n .
4. Given the set $X = \{1, \dots, m\}$ consider a “band” $R_n \subseteq X \times X$ around the diagonal of width $n \rightsquigarrow$ banded $m \times m$ matrices of band width n .

Operator systems associated to tolerance relations

- ▶ Key motivating example: a metric space (X, d) with the relation

$$\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$$

- ▶ If (X, μ) is a measure space and $\mathcal{R}_\epsilon \subseteq X \times X$ an open subset we obtain the operator system $E(\mathcal{R}_\epsilon)$ as the closure of integral operators with support in \mathcal{R}_ϵ . Note that $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$

Proposition

Let X be a complete, locally compact path metric measure space with a measure of full support. Then $C_{env}^(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$ and*

$$\text{prop}(E(\mathcal{R}_\epsilon)) = \lceil \text{diam}(X)/\epsilon \rceil$$

The pure states of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is ϵ -connected.

K-theory for operator systems

[arXiv:2409.02773]

A key invariant of C^* -algebras is K-theory. Is there an analogue for operator systems?

- ▶ Need notion of projection (*cf.* Araiza–Russell) or invertible selfadjoint elements
- ▶ It should capture the spectral localizer of Loring, Schulz-Baldes, and others
- ▶ It should be invariant under Morita equivalence [EKT]

Definition

A hermitian form x in a unital operator system E is a selfadjoint element $x \in M_n(E)$ which is non-degenerate in the sense that there exists $g > 0$ such that for all pure and maximal ucp maps $\phi : E \rightarrow B(\mathcal{H})$ we have

$$|\phi^{(n)}(x)| \geq g \cdot \text{id}_{\mathcal{H}}^{\oplus n}$$

In other words, x should have a gap g in each boundary representation

We will write $H(E, n)$ for all hermitian forms in $M_n(E)$.

Proposition

An element $x \in M_n(E)$ is non-degenerate if and only if $\iota_E^{(n)}(x)$ is an invertible element in the C^ -envelope $C_{env}^*(E)$.*

This is a consequence of the realization of the C^* -envelope in [Davidson–Kennedy]

Examples:

1. Hermitian forms (à la Witt) on a fgp right module pA^n over a C^* -algebra A : described by invertible elements $x = h + (1 - p) \in M_n(A)$ with $h \in pM_n(A)p$.
2. Projections p in operator systems à la Araiza–Russell are precisely projections in the C^* -envelope: $x = e - 2p$ is a hermitian form.
3. Similarly, ϵ -projections in quantitative K-theory define hermitian forms.
4. Spectral compressions of projections in C^* -algebra: $x = PYP$ with $Y = 1 - 2p$ provided $\|[P, p]\|$ sufficiently small.

The invariants and K-theory

$$\mathcal{V}(E, n) = H(E, n) / \sim_n$$

Example:

$$\mathcal{V}(\mathbb{C}, n) \cong \{-n, -n+2, \dots, n\}$$

and with the map $\iota_{nm}([x] = x \oplus e_{m-n}$ we have

$$\begin{array}{ccc} \mathcal{V}(\mathbb{C}, n) & \xrightarrow{\iota_{nm}} & \mathcal{V}(\mathbb{C}, m) \\ & \searrow \rho_n & \swarrow \rho_m \\ & \mathbb{Z} & \end{array}$$

In general, we consider

$$\mathcal{V}(E) = \varinjlim \mathcal{V}(E, n)$$

and $K_0(E)$ is the corresponding Grothendieck group (with identity $[e]$ and addition $'\oplus'$)

Properties of K_0

- ▶ For C^* -algebras we obtain usual K-theory via the map $[x] \mapsto [p = \frac{1}{2}(1 - x|x|^{-1})]$.
- ▶ Stability: we define a map $\iota_n : M_n(E) \rightarrow M_n(M_2(E))$ by

$$\iota_n(x) = \begin{pmatrix} x_{11} & 0 & x_{12} & 0 & \cdots & x_{1n} & 0 \\ 0 & e & 0 & 0 & \cdots & 0 & 0 \\ x_{21} & 0 & x_{22} & 0 & \cdots & \vdots & \\ 0 & 0 & 0 & e & \cdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ x_{n1} & 0 & \cdots & \cdots & \cdots & x_{nn} & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & e \end{pmatrix}$$

so that $\iota_n(x) \sim x$ (Whitehead). This allows to show $K_0(E) \cong K_0(M_2(E))$.

Non-unital operator systems and stability

The unitization [Werner, 2002] of a non-unital operator system E is given by the $*$ -vector space $E^+ = E \oplus \mathbb{C}$ with matrix order structure:

$$(x, A) \geq 0 \text{ iff } A \geq 0 \text{ and } \phi(A_\epsilon^{-1/2} x A_\epsilon^{-1/2}) \geq -1$$

for all $\epsilon > 0$ and noncommutative states $\phi \in \mathcal{S}_n(E)$, and where $A_\epsilon = \epsilon \mathbb{I}_n + A$.

$$\tilde{\mathcal{V}}(E, n) := \{(x, A) \in H(E^+, n) : A \sim_n \mathbb{I}_n\} / \sim_n$$

In the unital case, the isomorphism $E^+ \cong E \oplus \mathbb{C}$ given by $(x, A) \mapsto (x + Ae, A)$ yields that in this case

$$\tilde{\mathcal{V}}(E, n) \cong \mathcal{V}(E, n).$$

Theorem

For a unital operator system E we have $K_0(\mathcal{K} \otimes E) \cong K_0(E)$.

Stability

Theorem

For a unital operator system E we have $K_0(\mathcal{K} \otimes E) \cong K_0(E)$.

Proof.

1. Realize stabilization by maps $\kappa_{NM} : M_N(E) \rightarrow M_M(E)$, $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0_{M-N} \end{pmatrix}$
2. Commuting diagram:

$$\begin{array}{ccc} \tilde{\mathcal{V}}(E, n) & \xrightarrow{\kappa_{1N}} & \tilde{\mathcal{V}}(M_N(E), n) \\ \cong \downarrow & & \cong \downarrow \\ \mathcal{V}(E, n) & \xrightarrow[\cong]{\iota_n} & \mathcal{V}(M_N(E), n) \end{array}$$

3. The map $\kappa_{1\infty} : K_0(E) \rightarrow K_0(\mathcal{K} \otimes E)$ is an isomorphism:
 - ▶ injective: homotopy in $H((\mathcal{K} \otimes E)^+, n)$ compressed to homotopy in $H((M_N(E))^+, n)$.
 - ▶ surjective: approximation by finite-rank operators in norm is still hermitian form.



Summary

- ▶ Noncommutative geometry: metric aspect
- ▶ Operator systems: from (spectral) truncations and tolerance relations
- ▶ Duality of operator systems: state spaces
- ▶ New invariants: propagation number, K-theory
 - ▶ Higher K-group invariants [arXiv:2411.02981]:

$$\mathcal{V}_1^\delta(E, n) = \left\{ x \in M_n(E) : \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \text{ has spectral gap } \delta \right\} / \sim_n$$

and, more generally, $\mathcal{V}_p^\delta(E, n) := H^\delta(E \otimes \mathbb{C}l_p^{(1)}, n) / \sim_n$.

- ▶ Formal periodicity: $K_{2m}(E) = K_0(E)$ and $K_{2m+1}(E) = K_1(E)$.