A generalization of K-theory to operator systems

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Lorentz in October 1910

H.A. Lorentz by Jan Veth

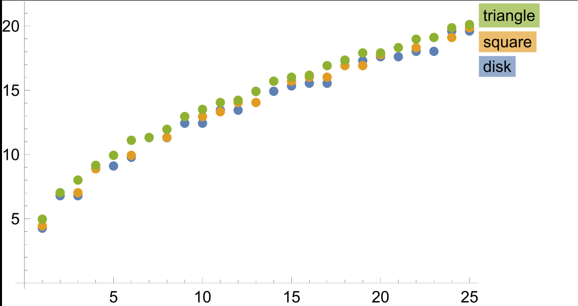
Origins of spectral geometry:

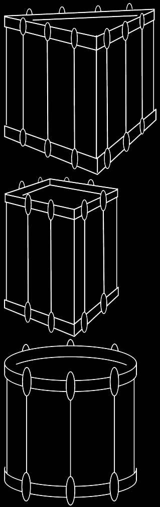
the high overtones behave inversely proportional to the volume.

Weyl in February 1911

$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$
$$\sim \frac{\Omega_d \mathrm{Vol}(M)}{d(2\pi)^d}\Lambda^d$$

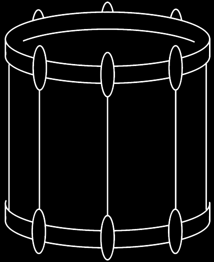
Evidence by the parabolic shapes ($\sqrt{\Lambda}$):





Mark Kac in 1966

"Can one hear the shape of a drum?"

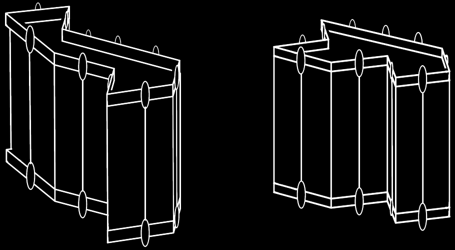


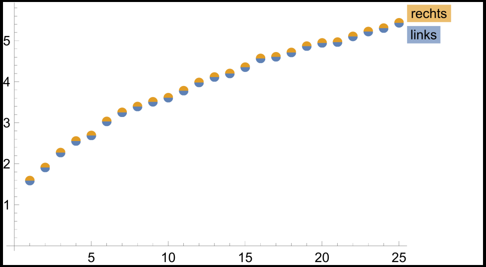
Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of M?

Isospectral drums!





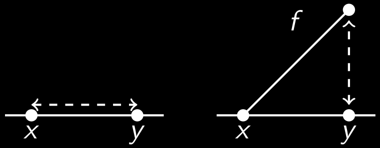
… so the answer to Kac's question is **no** and more information is needed…

Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

- Distance $d(x, y)$ between two points is usually defined as
  the **smallest** *of the arclengths (computed using the metric) of curves connecting x and y.*
- But it can also be defined as
  the **largest** *of differences $|f(x) - f(y)|$ for functions f with gradient $|\nabla f| \leq 1$.*

$$\boxed{d(x, y) = \sup_{\|[D_M, f]\| \leq 1} |\delta_x(f) - \delta_y(f)|}$$



Combination $(C^\infty(M), L^2(S_M), D_M)$ allows for reconstruction of geometry

Spectral triples

More generally, we consider a triple $(\mathcal{A}, \mathcal{H}, D)$

- a $*$-algebra $\mathcal{A}$
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A}$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ of norm 1
- Distance function on state space of $\mathcal{A}$:

$$d(\phi, \psi) = \sup_{a \in \mathcal{A}} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

Spectral data: $(\mathcal{A}, \mathcal{H}, D)$

- The mathematical reformulation of geometry in terms of spectral data (global analysis) requires the knowledge of the full Dirac operator.
- From a physical or computational standpoint this is not very realistic: detectors have limited energy ranges and resolution.
- We aim for the underlying mathematical formalism for doing (noncommutative) geometry with only part of the spectrum. [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged, ...)

Thus, we project onto part of the spectrum of D:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

Operator systems

Definition (Arveson 1969, Choi-Effros 1977)

An operator system is a $*$-closed vector space E of bounded operators. Unital: it contains the identity operator.

- E is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff

$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$

- in fact, E is matrix ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any n.

Maps between operator systems E, F are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

Isomorphisms are complete order isomorphisms

C^*-envelope of a unital operator system

Arveson introduced the notion of C^*-envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979. Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A C^*-extension $\kappa : E \to A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of a unital operator system is a C^*-extension $\kappa : E \to A$ with the following universal property:

$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & \searrow^{\lambda} & \uparrow\uparrow {\scriptstyle \exists!\rho} \\ & & B \end{array}$$

Example: operator system $C_{\text{harm}}(\overline{\mathbb{D}})$ of continuous harmonic functions with C^*-envelope $C(S^1)$.

Example: spectral truncation of the circle [Connes-vS, 2020]

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

- States are defined as unital positive linear functionals.

We have: $C^*_{\mathrm{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$

Gromov–Hausdorff convergence

Recall Gromov–Hausdorff distance between two metric spaces:

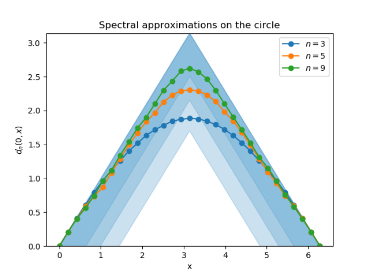
$$d_{\mathrm{GH}}(X, Y) = \inf\{d_H(f(X), g(Y)) \mid f : X \to Z, g : Y \to Z \text{ isometric}\}$$

and

$$d_H(X, Y) = \inf\{\epsilon \geq 0; X \subseteq Y_\epsilon, Y \subseteq X_\epsilon\}$$

Rieffel extends this to quantum metric spaces (essentially operator systems equipped with a Lip-norm).

Distance function for spectral truncations of the circle



Proposition (vS21, Hekkelman 2021)

The sequence of state spaces $\{(\mathcal{S}(P_n C(S^1) P_n), d_n)\}$ *converges to* $(\mathcal{S}(C(S^1)), d_{S^1})$ *in Gromov–Hausdorff distance.*

And more examples include (quantum) fuzzy spheres, Fourier truncations, truncations of tori (Leimbach–vS23, RU) ...

K-theory for operator systems

[arXiv:2409.02773]

A key invariant of C^*-algebras is K-theory. Is there an analogue for operator systems?

- Need notion of projection (*cf.* Araiza–Russell) or invertible selfadjoint elements
- It should capture the spectral localizer of Loring, Schulz-Baldes, and others
- It should be invariant under Morita equivalence [EKT]

Definition

A hermitian form *x in a unital operator system E is a selfadjoint element $x \in M_n(E)$ which is* non-degenerate *in the sense that $\imath_E^{(n)}(x)$ is an invertible element in the C^*-envelope $C^*_{\mathrm{env}}(E)$.*

We will write $H(E, n)$ for all hermitian forms in $M_n(E)$.

Examples:

1. Hermitian forms (à la Witt) on a fgp right module pA^n over a C^*-algebra A: described by invertible elements $x = h + (1 - p) \in M_n(A)$ with $h \in pM_n(A)p$.
2. Projections p in operator systems à la Araiza–Russell are precisely projections in the C^*-envelope: $x = e - 2p$ is a hermitian form.
3. Similarly, ϵ-projections in quantitative K-theory define hermitian forms.
4. Spectral compressions of projections in C^*-algebra: $x = PYP$ with $Y = 1 - 2p$ provided $\|[P, p]\|$ sufficiently small.

The invariants and K-theory

$$\boxed{\mathcal{V}(E,n) = H(E,n)/_{\sim_n}}$$

Example:

$$\mathcal{V}(\mathbb{C},n) \cong \{-n, -n+2, \ldots, n\}$$

and with the map $\imath_{nm}([x] = x \oplus e_{m-n}$ we have

$$\begin{array}{ccc} \mathcal{V}(\mathbb{C},n) & \xrightarrow{\quad \imath_{nm} \quad} & \mathcal{V}(\mathbb{C},m) \\ & {\scriptstyle \rho_n} \searrow \quad \swarrow {\scriptstyle \rho_m} & \\ & \mathbb{Z} & \end{array}$$

In general, we consider

$$\mathcal{V}(E) = \varinjlim \mathcal{V}(E,n)$$

and $K_0(E)$ is the corresponding Grothendieck group (with identity $[e]$ and addition $'\oplus'$)

Properties of K_0

- For C^*-algebras we obtain usual K-theory via the map $[x] \mapsto [p = \frac{1}{2}(1 - x|x|^{-1})]$.
- Stability: we define a map $\imath_n : M_n(E) \to M_n(M_2(E))$ by

$$\imath_n(x) = \left(\begin{array}{cc|cc|c|cc} x_{11} & 0 & x_{12} & 0 & \cdots & x_{1n} & 0 \\ 0 & e & 0 & 0 & & 0 & 0 \\ \hline x_{21} & 0 & x_{22} & 0 & \cdots & & \vdots \\ 0 & 0 & 0 & e & & & \\ \hline & \vdots & & \vdots & \ddots & & \vdots \\ \hline x_{n1} & 0 & & \cdots & \cdots & x_{nn} & 0 \\ 0 & 0 & & & & 0 & e \end{array}\right)$$

so that $\imath_n(x) \sim x$ (Whitehead). This allows to show $K_0(E) \cong K_0(M_2(E))$.

Non-unital operator systems

The unitization [Werner, 2002] of a non-unital operator system E is given by the $*$-vector space $E^+ = E \oplus \mathbb{C}$ with matrix order structure:

$$(x, A) \geq 0 \text{ iff } A \geq 0 \text{ and } \phi(A_\epsilon^{-1/2} x A_\epsilon^{-1/2}) \geq -1$$

for all $\epsilon > 0$ and noncommutative states $\phi \in \mathcal{S}_n(E)$, and where $A_\epsilon = \epsilon \mathbb{I}_n + A$.

$$\widetilde{\mathcal{V}}(E, n) := \left\{(x, A) \in H(E^+, n) : A \sim_n \mathbb{I}_n\right\} / _{\sim_n}$$

In the unital case, the isomorphism $E^+ \cong E \oplus \mathbb{C}$ given by $(x, A) \mapsto (x + Ae, A)$ yields that in this case

$$\widetilde{\mathcal{V}}(E, n) \cong \mathcal{V}(E, n).$$

Theorem

For a unital operator system E we have $K_0(\mathcal{K} \otimes E) \cong K_0(E)$.

Stability

Theorem

For a unital operator system E we have $K_0(\mathcal{K} \otimes E) \cong K_0(E)$.

Proof.

1. Realize stabilization by maps $\kappa_{NM} : M_N(E) \to M_M(E), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0_{M-N} \end{pmatrix}$
2. Commuting diagram:

$$\begin{array}{ccc} \widetilde{V}(E,n) & \xrightarrow{\kappa_{1N}} & \widetilde{\mathcal{V}}(M_N(E),n) \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{V}(E,n) & \xrightarrow[\cong]{\imath_n} & \mathcal{V}(M_N(E),n) \end{array}$$

3. The map $\kappa_{1\infty} : K_0(E) \to K_0(\mathcal{K} \otimes E)$ is an isomorphism:
   - injective: homotopy in $H((\mathcal{K} \otimes E)^+, n)$ compressed to homotopy in $H((M_N(E)^+, n)$.
   - surjective: approximation by finite-rank operators in norm is still hermitian form.

□

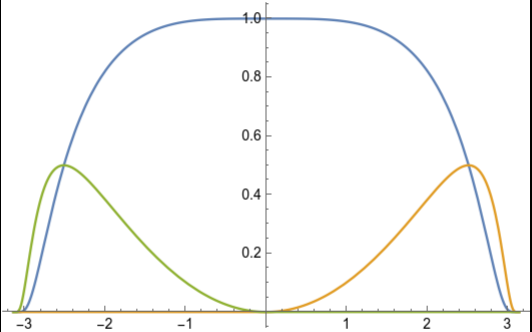
Example: spectral localizer on the 2-torus

Loring considered (in his thesis!) analogues of the Powers–Rieffel projections:

$$p = \begin{pmatrix} f & g + hU \\ g + hU^* & 1 - f \end{pmatrix} \in M_2(C^\infty(\mathbb{T}^2))$$

with U a unitary in the second variable, and f, g, h real-valued smooth functions in the first variable, satisfying

$$gh = 0, \qquad g^2 + h^2 = f - f^2.$$



Spectral truncations on $\mathbb{T}^2$

- We now consider spectral truncations $P = P_\rho$ onto $\ell^2\{\vec{n} \in \mathbb{Z}^2 : \|\vec{n}\| \leq \rho\} \subseteq l^2(\mathbb{Z}^2)$.
- We obtain a compression PYP of the hermitian form $Y = 1 - 2p$ on $\mathbb{T}^2$ corresponding to p:

$$PYP = \begin{pmatrix} P - 2PfP & -2PgP - 2PhUP \\ -2PgP - 2PhU^*P & -P + 2PfP \end{pmatrix} \in M_2(PC^\infty(\mathbb{T}^2)P)$$

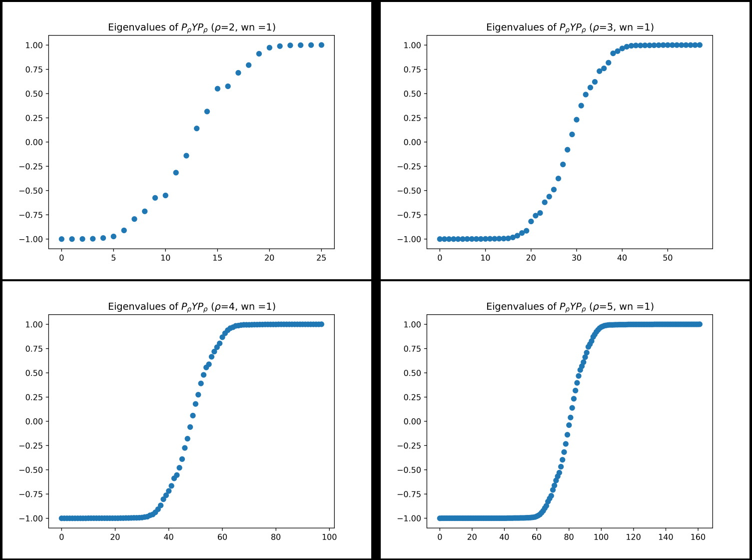
  For suitable P these are hermitian forms $\rightsquigarrow [PYP] \in K_0(PC(\mathbb{T}^2)P)$.
- The *spectral localizer* of Loring and Schulz-Baldes is given by the following matrix:

$$L_{\kappa,\rho} = \begin{pmatrix} -PYP & \kappa PD^+P \\ \kappa PD^-P & PYP \end{pmatrix}$$

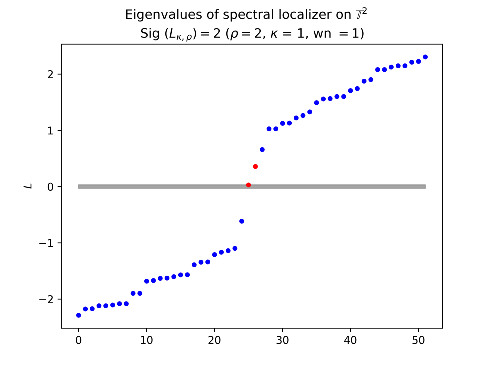
  In general they show that for suitable κ and ρ the index pairing can be computed as the signature of this matrix:

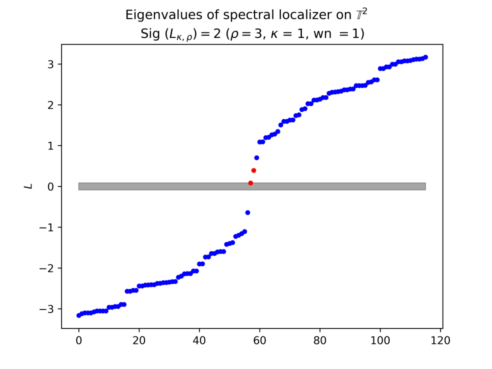
$$\boxed{\text{Index } pD^+p = \frac{1}{2}\text{Sig } L_{\kappa,\rho}}$$

Simulations: eigenvalues of PYP for $U(t_2) = e^{it_2}$

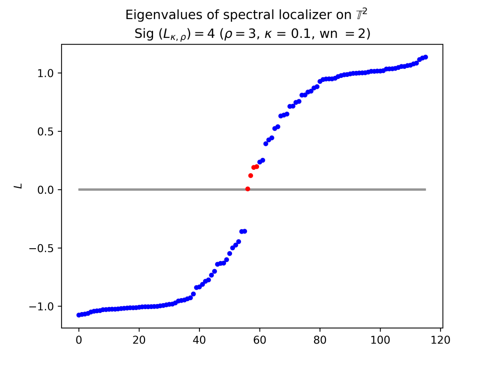


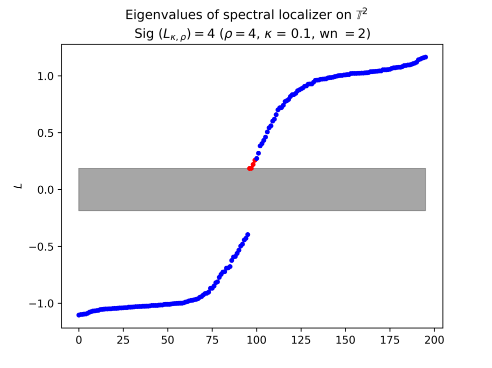
Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{it_2}$





Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{2it_2}$





Outlook

- Functoriality (ucp, cpc, order-zero,...)?
- Definition of higher K-groups [arXiv:2411.02981]:

$$\mathcal{V}_1^\delta(E,n) = \left\{ x \in M_n(E) : \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \text{ has spectral gap } \delta \right\} /_{\sim_n}$$

  and, more generally, $\mathcal{V}_p^\delta(E,n) := H^\delta(E \otimes \mathbb{C}l_p^{(1)}, n)/_{\sim_n}$.
- Formal periodicity: $K_{2m}(E) = K_0(E)$ and $K_{2m+1}(E) = K_1(E)$.
- Bott periodicity?