A generalization of K-theory to operator systems

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Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

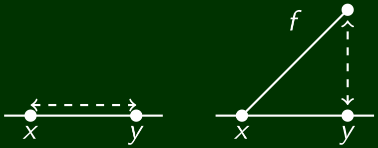
- Distance $d(x, y)$ between two points is usually defined as

  the **smallest** *of the arclengths (computed using the metric) of curves connecting x and y.*

- But it can also be defined as

  the **largest** *of differences $|f(x) - f(y)|$ for functions f with gradient $|\nabla f| \leq 1$.*

$$\boxed{d(x, y) = \sup_{\|[D_M, f]\| \leq 1} |\delta_x(f) - \delta_y(f)|}$$



Combination $(C^\infty(M), L^2(S_M), D_M)$
allows for reconstruction of geometry

Spectral triples

More generally, we consider a triple $(\mathcal{A}, \mathcal{H}, D)$

- a unital $*$-algebra $\mathcal{A}$
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A}$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ of norm 1
- Distance function on state space $S(\mathcal{A})$ of $\mathcal{A}$:

$$d_D(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$

These notions continue to make sense when we replace $\mathcal{A}$ by any self-adjoint vector space $\mathcal{E}$ of bounded operators on $\mathcal{H}$ that contains the unit, a so-called *operator system*.

Spectral data: $(\mathcal{A}, \mathcal{H}, D)$

- The mathematical reformulation of geometry in terms of spectral data (global analysis) requires the knowledge of the full Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.
- We aim for the underlying mathematical formalism for doing (noncommutative) geometry with only part of the spectrum and/or with finite resolution.

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged)

Operator systems

(I) Given $(A, \mathcal{H}, D)$ we project onto part of the spectrum of D:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

(II) Another approach would be to consider metric spaces up to a finite resolution :

- Consider integral operators associated to the tolerance relation R_ϵ given by $d(x, y) < \epsilon$

C^*-envelope of a unital operator system

[Arveson, 1969]

Hamana: existence and uniqueness in 1979; realized á la Arveson as direct sum of all boundary representations [Dritschel–McCullough 2005, Arveson 2008, Davidson–Kennedy 2015]

A C^*-extension $\kappa : E \to A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of a unital operator system is a C^*-extension $\kappa : E \to A$ with the following universal property:

$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & {\scriptstyle\lambda}\searrow & \uparrow{\scriptstyle \exists! \rho} \\ & & B \end{array}$$

Example: operator system $C_{\mathrm{harm}}(\overline{\mathbb{D}})$ of continuous harmonic functions with C^*-envelope $C(S^1)$.

How far is an operator system from a C^*-algebra?

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E.

Definition

The propagation number $\text{prop}(E)$ *of* E *is defined as the smallest integer* n *such that* $(\imath_E(E))^{\circ n} \subseteq C^*_{env}(E)$ *is a* C^**-algebra.*

Returning to the harmonic functions in the disk we have $\text{prop}(C_{\text{harm}}(\overline{\mathbb{D}})) = 1$.

Proposition (Connes-vS, 2020; Pawlowska, 2024)

The propagation number is invariant under complete order isomorphisms, as well as under stable=Morita equivalence [EKT, 2019]:

$$prop(E) = prop(E \otimes_{min} \mathcal{K})$$

More generally [Koot, 2021], we have

$$prop(E \otimes_{\text{min}} F) = \max\{prop(E), prop(F)\}$$

Example: spectral truncation of the circle [Connes-vS, 2020]

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\text{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is the operator system of Toeplitz matrices:

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

- States are defined as unital positive linear functionals.

We have: $C^*_{\text{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\text{prop}(C(S^1)^{(n)}) = 2$ (for any n).

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.
- The C^*-envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z}) \cong C(S^1)$; propagation number ∞

Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. *The extreme rays in* $C(S^1)_+^{(n)}$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n / S_n$.

Operator systems, groupoids and bonds (aka a positivity domain)

Definition (Connes-vS, 2021)

A bond is a triple (G, ν, Ω) consisting of a locally compact groupoid G, a Haar system $\nu = \{\nu_x\}$ and an open symmetric subset $\Omega \subseteq G$ containing the units $G^{(0)}$.

Proposition

Let (Ω, G, ν) be a bond. The closure of the subspace $C_c(\Omega) \subseteq C_c(G)$ in the C^*-algebra $C^*(G)$ is a (possibly non-unital) operator system.

Example

1. *Consider Ω in a l.c. Lie group $G \rightsquigarrow$ Fourier truncations (á la Rieffel) in $C^*(G)$*
2. *Consider $\Omega_n = (-n, n) \subset \mathbb{Z} \rightsquigarrow$ Fejér–Riesz system $C^*(\mathbb{Z})_{(n)} \cong (C(S^1)^{(n)})^d$.*
3. *Consider $\Omega_n = (-n, n) \subseteq C_m$ (so modulo m). The operator system consists of banded $m \times m$ circulant matrices of band width n.*
4. *Given the set $X = \{1, \ldots, m\}$ consider a "band" $R_n \subseteq X \times X$ around the diagonal of width $n \rightsquigarrow$ banded $m \times m$ matrices of band width n.*

Operator systems associated to tolerance relations

- Key motivating example: a metric space (X, d) with the relation

$$\boxed{\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}}$$

- If (X, μ) is a measure space and $\mathcal{R}_\epsilon \subseteq X \times X$ an open subset we obtain the operator system $E(\mathcal{R}_\epsilon)$ as the closure of integral operators with support in $\mathcal{R}_\epsilon$. Note that $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$

Proposition

Let X be a complete, locally compact path metric measure space with a measure of full support. Then $C^*_{env}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$ and

$$\mathrm{prop}(E(\mathcal{R}_\epsilon)) = \lceil diam(X)/\epsilon \rceil$$

The pure states of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is ϵ-connected.

K-theory for operator systems

[arXiv:2409.02773]

A key invariant of C^*-algebras is K-theory. Is there an analogue for operator systems?

- Need notion of projection (*cf.* Araiza–Russell) or invertible selfadjoint elements
- It should capture the spectral localizer of Loring, Schulz-Baldes, and others
- It should be invariant under Morita equivalence [EKT]

Definition

A hermitian form x *in a unital operator system* E *is a selfadjoint element* $x \in M_n(E)$ *which is* non-degenerate *in the sense that there exists* $g > 0$ *such that for all pure and maximal ucp maps* $\phi : E \to B(\mathcal{H})$ *we have*

$$|\phi^{(n)}(x)| \geq g \cdot \mathrm{id}_{\mathcal{H}}^{\oplus n}$$

In other words, x *should have a gap* g *in each boundary representation*

We will write $H(E, n)$ for all hermitian forms in $M_n(E)$.

Proposition

An element $x \in M_n(E)$ is non-degenerate if and only if $\imath_E^{(n)}(x)$ is an invertible element in the C^*-envelope $C^*_{env}(E)$.

This is a consequence of the realization of the C^*-envelope in [Davidson–Kennedy]

Examples:

1. Hermitian forms (à la Witt) on a fgp right module pA^n over a C^*-algebra A: described by invertible elements $x = h + (1 - p) \in M_n(A)$ with $h \in pM_n(A)p$.
2. Projections p in operator systems à la Araiza–Russell are precisely projections in the C^*-envelope: $x = e - 2p$ is a hermitian form.
3. Similarly, ϵ-projections in quantitative K-theory define hermitian forms.
4. Spectral compressions of projections in C^*-algebra: $x = PYP$ with $Y = 1 - 2p$ provided $\|[P, p]\|$ sufficiently small.

The invariants and K-theory

$$\boxed{\mathcal{V}(E, n) = H(E, n)/_{\sim_n}}$$

Example:

$$\mathcal{V}(\mathbb{C}, n) \cong \{-n, -n+2, \ldots, n\}$$

and with the map $\imath_{nm}([x] = x \oplus e_{m-n}$ we have

$$\begin{array}{ccc} \mathcal{V}(\mathbb{C}, n) & \xrightarrow{\imath_{nm}} & \mathcal{V}(\mathbb{C}, m) \\ & {\scriptstyle \rho_n} \searrow \quad \swarrow {\scriptstyle \rho_m} & \\ & \mathbb{Z} & \end{array}$$

In general, we consider

$$\mathcal{V}(E) = \varinjlim \mathcal{V}(E, n)$$

and $K_0(E)$ is the corresponding Grothendieck group (with identity $[e]$ and addition $'\oplus'$)

Properties of K_0

- For C^*-algebras we obtain usual K-theory via the map $[x] \mapsto [p = \frac{1}{2}(1 - x|x|^{-1})]$.
- Stability: we define a map $\imath_n : M_n(E) \to M_n(M_2(E))$ by

$$\imath_n(x) = \left(\begin{array}{cc|cc|c|cc} x_{11} & 0 & x_{12} & 0 & \dots & x_{1n} & 0 \\ 0 & e & 0 & 0 & & 0 & 0 \\ \hline x_{21} & 0 & x_{22} & 0 & \dots & \multicolumn{2}{c}{\vdots} \\ 0 & 0 & 0 & e & & & \\ \hline \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c}{\vdots} \\ \hline x_{n1} & 0 & \multicolumn{2}{c|}{\dots} & \dots & x_{nn} & 0 \\ 0 & 0 & & & & 0 & e \end{array}\right)$$

so that $\imath_n(x) \sim x$ (Whitehead). This allows to show $K_0(E) \cong K_0(M_2(E))$.

Non-unital operator systems and stability

The unitization [Werner, 2002] of a non-unital operator system E is given by the $*$-vector space $E^+ = E \oplus \mathbb{C}$ with matrix order structure:

$$(x, A) \geq 0 \text{ iff } A \geq 0 \text{ and } \phi(A_\epsilon^{-1/2} x A_\epsilon^{-1/2}) \geq -1$$

for all $\epsilon > 0$ and noncommutative states $\phi \in \mathcal{S}_n(E)$, and where $A_\epsilon = \epsilon \mathbb{I}_n + A$.

$$\widetilde{\mathcal{V}}(E, n) := \left\{(x, A) \in H(E^+, n) : A \sim_n \mathbb{I}_n\right\} / _{\sim_n}$$

In the unital case, the isomorphism $E^+ \cong E \oplus \mathbb{C}$ given by $(x, A) \mapsto (x + Ae, A)$ yields that in this case

$$\widetilde{\mathcal{V}}(E, n) \cong \mathcal{V}(E, n).$$

Theorem

For a unital operator system E we have $K_0(\mathcal{K} \otimes E) \cong K_0(E)$.

Stability

Theorem

For a unital operator system E we have $K_0(\mathcal{K} \otimes E) \cong K_0(E)$.

Proof.

1. Realize stabilization by maps $\kappa_{NM} : M_N(E) \to M_M(E), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0_{M-N} \end{pmatrix}$
2. Commuting diagram:

$$\begin{array}{ccc} \widetilde{V}(E,n) & \xrightarrow{\kappa_{1N}} & \widetilde{\mathcal{V}}(M_N(E),n) \\ \cong \downarrow & & \cong \downarrow \\ \mathcal{V}(E,n) & \xrightarrow[\cong]{\imath_n} & \mathcal{V}(M_N(E),n) \end{array}$$

3. The map $\kappa_{1\infty} : K_0(E) \to K_0(\mathcal{K} \otimes E)$ is an isomorphism:
   - injective: homotopy in $H((\mathcal{K} \otimes E)^+, n)$ compressed to homotopy in $H((M_N(E)^+, n)$.
   - surjective: approximation by finite-rank operators in norm is still hermitian form.

□

Summary

- Noncommutative geometry: metric aspect
- Operator systems: from (spectral) truncations and tolerance relations
- Duality of operator systems: state spaces
- New invariants: propagation number, K-theory
  - Higher K-group invariants [arXiv:2411.02981]:

$$\mathcal{V}_1^\delta(E,n) = \left\{ x \in M_n(E) : \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \text{ has spectral gap } \delta \right\} / \sim_n$$

    and, more generally, $\mathcal{V}_p^\delta(E,n) := H^\delta(E \otimes \mathbb{C}l_p^{(1)}, n)/\sim_n$.
  - Formal periodicity: $K_{2m}(E) = K_0(E)$ and $K_{2m+1}(E) = K_1(E)$.