Second quantization of noncommutative spaces: emergence of the Standard Model and gravity

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A fermion in spacetime

Minimal ingredients to describe a free fermion:

- coordinates on spacetime M:

$$x_\mu \cdot x_\nu(p) = x_\mu(p) x_\nu(p), etc.,$$

- propagation, described by Dirac operator $D_M = i\gamma^\mu \partial_\mu$

Noncommutative geometry

- Combination of coordinate algebra and operators is central to the noncommutative approach [Connes 1994], in terms of spectral triples:

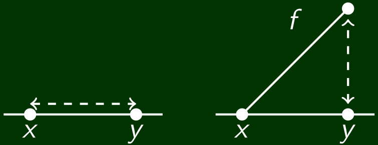
$$(\mathcal{A}, \mathcal{H}, D)$$

- The commutative case (Riemannian spin manifold M):
  - the algebra $C^\infty(M)$ of smooth functions on M
  - the Dirac operator D_M
  - both acting on Hilbert space $L^2(S_M)$ of square-integrable spinors.
- The noncommutative case:
  - an algebra $\mathcal{A}$
  - a (suitable) self-adjoint operator D
  - both acting on a Hilbert space $\mathcal{H}$

Reconstruction of geometry

- Reconstruction of M in the commutative case [Connes 1989]: $(C^\infty(M), L^2(\mathcal{S}_M), D_M)$:

$$d(x,y) = \sup_f \{|f(x) - f(y)| : \text{ gradient } f \leq 1\}$$



- The gradient of f is given by the (norm of the) commutator $[D_M, f] = D_M f - f D_M$ (*e.g.* $[D_\mathbb{R}, f] = -i\frac{df}{dt}$)

Emerging bosons

Our fermionic starting point induces a bosonic theory:

- "Inner perturbations" by the coordinates [C 1996, CCS 2013]:

  $$D_M \rightsquigarrow D_M + \sum_j a_j [D_M, a_j']$$

  for functions a_j, a_j' depending on the coordinates x_μ.
- Then,

  $$\sum_j a_j [D_M, a_j'] = A^\nu \gamma^\mu (\partial_\mu x^\nu) = A^\mu \gamma_\mu$$

  where A^μ is the electromagnetic 4-potential describing the photon.

Entering **noncommutativity**

Consider a finite space F, but with a *noncommutative* structure:

- Described by block diagonal matrices ("noncommutative coordinates")

  $$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & a_N \end{pmatrix},$$

  where the $a_1, a_2, \ldots, a_N$ are square matrices of size $n_1, n_2, \ldots, n_N$.
- Hence we will consider the matrix algebra

  $$A_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C})$$

  where $\mathbb{C}$ can be replaced by $\mathbb{R}$ or $\mathbb{H}$.
- A finite Dirac operator is given by a hermitian matrix.

Example: **commutative** two-point space

$$F = {}_1\bullet \qquad\qquad {}_2\bullet$$

- Then the algebra of smooth functions

$$C^\infty(F) := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

- A finite Dirac operator is given by

$$D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}; \qquad (c \in \mathbb{C})$$

- The distance formula then becomes

$$d(1,2) = \max \left\{ |\lambda_1 - \lambda_2| : \left\| \begin{pmatrix} 0 & \overline{c}(\lambda_2 - \lambda_1) \\ c(\lambda_1 - \lambda_2) & 0 \end{pmatrix} \right\| \leq 1 \right\} = \frac{1}{|c|}$$

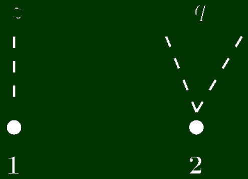
Example: noncommutative two-point space

Coordinates on F are elements in $\mathbb{C} \oplus \mathbb{H}$

- A complex number z
- A quaternion $q = q_0 + iq_k\sigma^k$; in terms of Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It describes a two-point space, with internal structure:



Inner perturbations on nc two-point space

- 'Dirac operator' $D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- Inner perturbations:

$$D_F \rightsquigarrow D_F + \sum_j a_j [D_F, a'_j] = \begin{pmatrix} 0 & \overline{c\phi_1} & \overline{c\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$

- Distance between the two points is now $1/\sqrt{|c\phi_1|^2 + |c\phi_2|^2}$.
- We may call ϕ_1 and ϕ_2 the Higgs field.
- Indeed, the group of unitary block diagonal matrices is now $U(1) \times SU(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Almost-commutative spacetimes

We now combine mild matrix noncommutativity with spacetime:

- coordinates of the almost-commutative spacetime $M \times F$:

$$\boxed{\hat{x}^\mu(p) = (z^\mu(p), q^\mu(p))}$$

  as elements in $\mathbb{C} \oplus \mathbb{H}$ (for each μ and each point p of M)

- The combined Dirac operator becomes

$$\boxed{D_{M\times F} = D_M + \gamma_5 D_F}$$

  Note that $D_{M\times F}^2 = D_M^2 + D_F^2$, which will be useful later on.

Inner perturbations on $M \times F$

So, we describe $M \times F$ by:

$$\hat{x}^{\mu} = (z^{\mu}, q^{\mu})\,; \qquad D_{M\times F} = D_M + \gamma_5 D_F$$

As before, we consider inner perturbations of $D_{M\times F}$ by $\hat{x}^{\mu}(p)$:

- The inner perturbations of D_F become scalar fields ϕ_1, ϕ_2.
- The inner perturbations of D_M become matrix-valued:

$$\sum_j a_j[D_M, a_j'] = a_{\nu}\gamma^{\mu}(\partial_{\mu}\hat{x}^{\nu}) =: A_{\mu}\gamma^{\mu}$$

with A_{μ} taking values in $\mathbb{C} \oplus \mathbb{H}$:

$$A_{\mu} = \begin{pmatrix} B_{\mu} & 0 & 0 \\ 0 & W_{\mu}^{3} & W_{\mu}^{+} \\ 0 & W_{\mu}^{-} & -W_{\mu}^{3} \end{pmatrix}$$

corresponding to hypercharge and the W-bosons.

What do we have so far?

- A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ allows to write the Dirac equation for $\psi \in \mathcal{H}$:

$$\boxed{D\psi = 0}$$

- This describes the propagation of a single fermion (described by the 'wave function' ψ): one-particle quantum physics.
- We keep this first quantization as a mystery, and will consider second quantization as a functor (free on Edward Nelson).
- Let us start with some basic theory on second quantization [Araki 1987, Carey 1987, GVF 2001]

Clifford algebra: from $\mathcal{H} \to \mathbb{C}\mathrm{l}(\mathcal{H}_{\mathbb{R}})$

- Given a Hilbert space $\mathcal{H}$, consider the real Euclidean vector space underlying it: $V = \mathcal{H}_{\mathbb{R}}$.
- In finite dimensions, we may define $\mathbb{C}\mathrm{l}(V) = \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$ as the complexified Clifford algebra, generated by $c(v)$ for any $v \in V$ with relations:

$$c(v)c(w) + c(w)c(v) = 2\langle v, w\rangle$$

- In the infinite-dimensional case, we consider the algebraic Clifford algebra $\mathbb{C}\mathrm{l}^{\mathrm{alg}}(V) = \cup \mathbb{C}\mathrm{l}(V_f)$ with the union over all finite-dimensional subspaces $V_f \subseteq V$.
- This $*$-algebra is equipped with a trace τ such that $\tau(1) = 1$, we define $\mathbb{C}\mathrm{l}(V)$ as the C^*-closure in $B(H_\tau)$ for the Hilbert space completion H_τ of $\mathbb{C}\mathrm{l}^{\mathrm{alg}}(V)$. [Plymen–Robinson, 1994]

C^*-dynamical system and KMS-condition

- The self-adjoint operator D in $\mathcal{H}$ is used as the generator of a one-parameter group of automorphisms of the Clifford algebra:

$$\sigma_t^D(c(v)) = c(e^{itD}v); \qquad (v \in V_f)$$

- In short the KMS condition at inverse temperature β means that one has the formal equality

$$\varphi(a\sigma_t(b))|_{t=i\beta} = \varphi(ba) \qquad a, b \in \mathcal{C}.$$

Proposition

For any $\beta > 0$ there exists a unique KMS$_\beta$ state φ_β on the C^*-dynamical system $(\mathbb{C}\mathrm{l}(\mathcal{H}_\mathbb{R}), \sigma_t^D)$.

Proposition

If the operator $\exp(-\beta|D|)$ is of trace class, the state φ_β is of type I and the associated irreducible representation is given by the fermionic second quantization associated to the complex structure $I := i\,\mathrm{sign}\,D$ on $\mathcal{H}_\mathbb{R}$.

Fermionic second quantization

- Equip $\mathcal{H}_{\mathbb{R}}$ with complex structure, *e.g.* $I = i\,\mathrm{sign} D \rightsquigarrow$ Dirac sea e^{ItD}
- $\mathbb{C}\mathrm{l}(\mathcal{H}_{\mathbb{R}})$ acts on the Fock space $\bigwedge \mathcal{H}_I$ via

$$\gamma_I(v) = a_I^*(v) + a_I(v); \qquad (v \in \mathcal{H}_{\mathbb{R}}).$$

Proposition (Chamseddine–Connes–vS, 2018)

(*i*) *The one-parameter group* σ_t^D *is implemented in the (physical) Fock representation by the one-parameter unitary group* $\bigwedge \exp(it|D|)$*:*

$$\gamma_I(\sigma_t^D(A)) = \bigwedge(e^{it|D|})\gamma_I(A)\bigwedge(e^{-it|D|}) \qquad A \in \mathbb{C}\mathrm{l}(\mathcal{H}_{\mathbb{R}}).$$

(*ii*) *If* $\exp(-\beta|D|)$ *is of trace class the state* φ_β *is of type I and is given by*

$$\varphi_\beta(A) = \mathcal{N}^{-1}\,\mathrm{Trace}\left(\bigwedge \exp(-\beta|D|)\gamma_I(A)\right) \qquad A \in \mathbb{C}\mathrm{l}(\mathcal{H}_{\mathbb{R}})$$

Gibbs states and entropy

- We thus have a density matrix

$$\rho_\beta = \mathcal{N}^{-1} \cdot \bigwedge(e^{-\beta|D|})$$

- Note that this is the Gibbs state for a Fermi gas on the (noncommutative) space that is described by $(\mathcal{A}, \mathcal{H}, D)$.

Theorem (Chamseddine–Connes–vS, 2018)

The (von Neumann) entropy,

$$S(\rho_\beta) = -\operatorname{Trace} \rho_\beta \log \rho_\beta,$$

*of the above Gibbs state ρ_β is given by a spectral action $\operatorname{Trace}_{\mathcal{H}} h(\beta D)$ for the function $h(x) = \mathcal{E}(e^{-x})$ where $\mathcal{E}(y)$ is the entropy of a partition of the unit interval in two intervals with size of ratio y (*i.e.* of size $1/(1+y)$ and $y/(1+y)$).*

Analysis of the function h

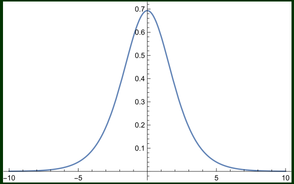
- $\mathcal{E}(y)$ is the entropy of a partition of the unit interval in two intervals with size of ratio y:

$$\mathcal{E}(y) = -\operatorname{Trace} \rho_y \log \rho_y; \qquad \rho_y = \begin{pmatrix} \frac{1}{1+y} & 0 \\ 0 & \frac{y}{1+y} \end{pmatrix}.$$

- We have $\mathcal{E}(y) = \log(y+1) - \frac{y \log y}{y+1}$

$$h(x) = \mathcal{E}(e^{-x}) = \frac{x}{1+e^x} + \log(1+e^{-x})$$

and this is applied to the spectrum of βD.



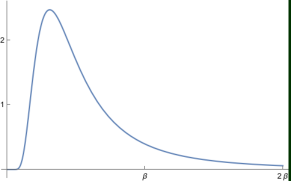
Example: entropy of two-point space

$$F = \; {}_1\bullet \qquad\qquad {}_2\bullet$$

- Distance $r := d(1,2) = 1/|c|$ in terms of $D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}$.
- For $r \to 0$ we have $S(\rho_\beta) = 0$;
- For $r \to \infty$ we have maximum entropy $S(\rho_\beta) = 2\log 2$;

Entropic force $F(r) = \beta^{-1}\partial_r S(\rho_\beta)$?

$$F(r) = \frac{\beta^3/2r^3}{\cosh^2(\beta/2r)}$$



Laplace transform and heat expansion

Proposition (Chamseddine–Connes–vS, 2018)

The function h is a Laplace transform:

$$h(x) = \int_0^\infty g(t) e^{-tx^2}$$

with

$$g(t) = \frac{-1}{8\sqrt{\pi} t^{5/2}} \sum_{n\in\mathbb{Z}} (-1)^n n^2 q^{n^2}; \qquad q = e^{-1/4t}.$$

This allows us to use heat asymptotics of $e^{-t\beta^2 D^2}$ to determine Trace $h(\beta D)$.

Asymptotic expansion of entropy

If $\operatorname{Trace} e^{-tD^2} \sim \sum_k t^k b_k$ then

$$S(\rho_\beta) = \operatorname{Trace} h(\beta D) \sim \sum_k \beta^{2k} \gamma(k) b_k \qquad \gamma(k) = \frac{1-2^{-2k}}{k} \pi^{-k} \xi(2k)$$

in terms of the Riemann ξ-function :

$$\xi(s) := \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$$

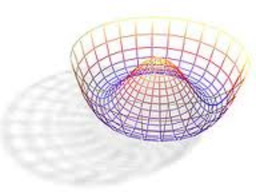
$\gamma(-1)$	$\gamma(-1/2)$	$\gamma(0)$	$\gamma(1/2)$	$\gamma(1)$	$\gamma(3/2)$
$\frac{9\zeta(3)}{2}$	$\frac{\pi^{3/2}}{3}$	$\log 2$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{8}$	$\frac{7\zeta(3)}{8\pi^{5/2}}$

Example: entropy of the electroweak theory

Use $D^2_{M\times F} = D^2_M + D^2_F$ with noncommutative two-point space $F = \mathbb{C}\oplus\mathbb{H}$ to compute (in 4d)

$$S(\rho_\beta) = \mathrm{Trace}\, h(\beta D_{M\times F}) \sim c_4\beta^{-4}\mathrm{Vol}(M) + c_2\beta^{-2}\int R\sqrt{g}$$
$$- c_0 \int C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \int c_0'\,\mathrm{Trace}\, F_{\mu\nu}F^{\mu\nu} - c_2'\beta^{-2}|\phi|^2 + c_0'|\phi|^4 + \cdots$$

- (Higher-derivative) gravity
- The Yang–Mills term $F_{\mu\nu}F^{\mu\nu}$ for hypercharge and W-boson
- The Higgs potential $-\mu^2|\phi|^2 + \lambda|\phi|^4$



Beyond the SM with noncommutative geometry

[Chamseddine–Connes–vS, 2013, 2015]

- The matrix coordinates of the Standard Model in $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ arise naturally as a restriction of the following coordinates

  $$\hat{x}^\mu(p) = \left(q_R^\mu(p), q_L^\mu(p), m^\mu(p)\right) \in \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

  corresponding to a Pati–Salam unification:

  $$U(1)_Y \times SU(2)_L \times SU(3) \to SU(2)_R \times SU(2)_L \times SU(4)$$

- The 96 fermionic degrees of freedom are structured as

  $$\left(\begin{array}{cc|cc} \nu_R & u_{iR} & \nu_L & u_{iL} \\ e_R & d_{iR} & e_L & d_{iL} \end{array} \right) \qquad (i = 1, 2, 3)$$

- The finite Dirac operator is a 96 × 96-dimensional matrix containing Yukawa mass matrices, etc.

Inner perturbations

- Inner perturbations of D_M now give three gauge bosons:

  $$W_R^\mu, \qquad W_L^\mu, \qquad V^\mu$$

  corresponding to $SU(2)_R \times SU(2)_L \times SU(4)$.
- For the inner perturbations of D_F we distinguish two cases, depending on the initial form of D_F:
  - I The Standard Model $D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$
  - II A more general D_F with zero $\overline{f}_L - f_L$-interactions.

Scalar sector of the spectral Pati–Salam model

I For a SM D_F, the resulting scalar fields are composite fields, expressed in scalar fields whose representations are:

	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\phi^b_{\dot a}$	2	2	1
$\Delta_{\dot a I}$	2	1	4
Σ^I_J	1	1	15

II For a more general finite Dirac operator, we have fundamental scalar fields:

particle	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\Sigma^{bJ}_{\dot a J}$	2	2	$1 + 15$
$H_{\dot a I \dot b J}$	3	1	10
	1	1	6

A dictionary and outlook

one-particle	second-quantized
$\mathcal{A}$	$\sigma_t^D \mapsto \sigma_t^{D_A}$
$\mathcal{H}$	$\mathbb{C}\mathrm{l}(\mathcal{H}_{\mathbb{R}})$
D	$\{\sigma_t^D\}_t$ arising from e^{itD}
spectral action	entropy of KMS

- Geometric significance of this entropy
- Extension to type II?
- Quantization of inner perturbations