Operator systems and spectral truncations

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Lorentz in October 1910

H.A. Lorentz by Jan Veth

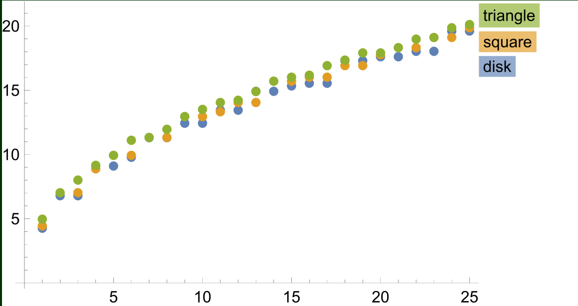
Origins of spectral geometry:

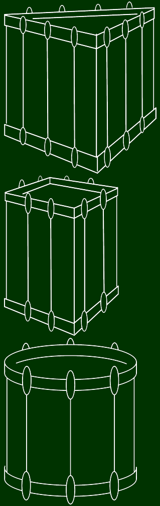
"Hierbei entseht das mathematische Problem, zu beweisen, dass die Anzahl der genügend hohen Obertöne zwischen n und $n + dn$ unabhängig von der Gestalt der Hülle und nur ihrem Volumen proportional ist."

Weyl in February 1911

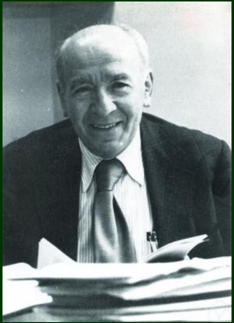
$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$
$$\sim \frac{\Omega_d \mathrm{Vol}(M)}{d(2\pi)^d} \Lambda^d$$

Evidence by the parabolic shapes ($\sqrt{\Lambda}$):

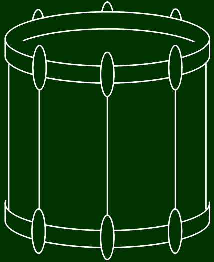




Mark Kac in 1966



"Can one hear the shape of a drum?"

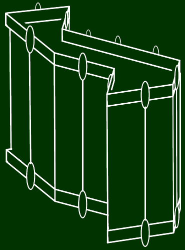


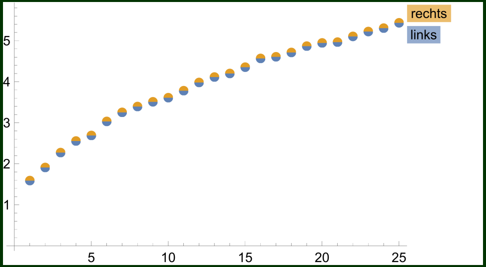
Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of M?

Isospectral drums!



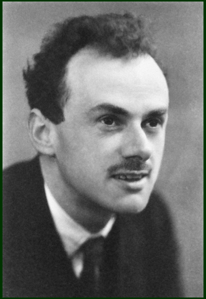


… so the answer to Kac's question is **no**
and more information is needed…

Analysis: Dirac operator

Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator is a ‘square-root’ of the Laplacian, so that its spectrum give the wave numbers k.
- First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold M.



The circle

- The Laplacian on the circle $\mathbb{S}^1$ is given by

$$\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}; \qquad (t \in [0, 2\pi))$$

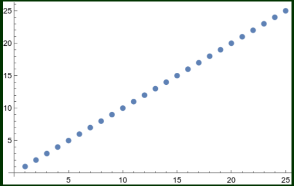
- The Dirac operator on the circle is

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

  with square $\Delta_{\mathbb{S}^1}$.

- The eigenfunctions of $D_{\mathbb{S}^1}$ in $L^2(S^1)$ are the complex exponential functions

$$e^{int} = \cos nt + i \sin nt; \qquad (n \in \mathbb{Z})$$



and $[D_{S^1}, f] = \frac{df}{dt}$, a bounded operator on $L^2(S^1)$ for smooth f.

The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- The Laplacian reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

- Dirac suggested to consider operators of the form $D_{\mathbb{T}^2} = a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2}$ with complex *matrices* as coefficients:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

then $a^2 = b^2 = -1$ and $ab + ba = 0$

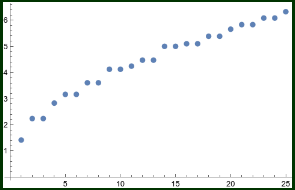
- The Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

which satisfies $(D_{\mathbb{T}^2})^2 = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}$.

- The spectrum of the Dirac operator $D_{\mathbb{T}^2}$ is

$$\left\{ \pm\sqrt{n_1^2 + n_2^2} : n_1, n_2 \in \mathbb{Z} \right\};$$



and $\|[D_{\mathbb{T}^2}, f]\| = \|f\|_{\mathrm{Lip}}$.

More generally, a Dirac operator exists on spin manifolds as a differential operator acting in $L^2(S_M)$ and square $D_M^2 = \Delta_M + \frac{1}{4}\kappa$ for which furthermore

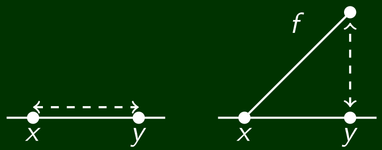
$$\|[D_M, f]\| = \|f\|_{\mathrm{Lip}}$$

Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

- Distance $d(x,y)$ between two points is usually defined as
  the **smallest** *of the arclengths (computed using the metric) of curves connecting x and y.*
- But it can also be defined as
  the **largest** *of differences $|f(x) - f(y)|$ for functions f with gradient $|\nabla f| \leq 1$.*

$$d(x,y) = \sup_{\|[D_M, f]\| \leq 1} |\delta_x(f) - \delta_y(f)|$$



Combination $(C^\infty(M), L^2(S_M), D_M)$
allows for reconstruction of geometry

Spectral triples

More generally, we consider a triple $(\mathcal{A}, \mathcal{H}, D)$

- a unital $*$-algebra $\mathcal{A}$
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A}$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ of norm 1
- Distance function on state space $S(\mathcal{A})$ of $\mathcal{A}$:

$$d_D(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$

These notions continue to make sense when we replace $\mathcal{A}$ by any self-adjoint vector space $\mathcal{E}$ of bounded operators on $\mathcal{H}$ that contains the unit, a so-called *operator system*.

Spectral data: $(\mathcal{A}, \mathcal{H}, D)$

- The mathematical reformulation of geometry in terms of spectral data (global analysis) requires the knowledge of the full Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.
- We aim for the underlying mathematical formalism for doing (noncommutative) geometry with only part of the spectrum and/or with finite resolution.

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged)

Operator systems

(I) Given $(A, \mathcal{H}, D)$ we project onto part of the spectrum of D:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

(II) Another approach would be to consider metric spaces up to a finite resolution :

- Consider integral operators associated to the tolerance relation R_ϵ given by $d(x, y) < \epsilon$

Preliminaries on operator systems

Definition

We say that a $*$-vector space is matrix ordered *if*

1. *for each n we are given a cone of positive elements $M_n(E)_+$ in $M_n(E)_h$,*
2. $M_n(E)_+ \cap (-M_n(E)_+) = \{0\}$ *for all n,*
3. *for every m, n and $A \in M_{mn}(\mathbb{C})$ we have that $AM_n(E)_+A^* \subseteq M_m(E)_+$.*

We call $e \in E_h$ an *order unit* for E if for each $x \in E_h$ there is a $t > 0$ such that $-te \leq x \leq te$. It is called an *Archimedean order unit* if $-te \leq x$ for all $t > 0$ implies that $x \geq 0$.

Definition

An (abstract) operator system *is given by a matrix-ordered $*$-vector space E with an order unit e such that for all n $e^{\oplus n}$ is an Archimedean order unit for $M_n(E)$.*

Maps between operator systems E, F are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

Isomorphisms are complete order isomorphisms

C^*-envelope of a unital operator system

[Arveson, 1969]

Hamana: existence and uniqueness in 1979; realized á la Arveson as direct sum of all boundary representations [Dritschel–McCullough 2005, Arveson 2008, Davidson–Kennedy 2015]

A C^*-extension $\kappa : E \to A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of a unital operator system is a C^*-extension $\kappa : E \to A$ with the following universal property:

$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & \underset{\lambda}{\searrow} & \Big\uparrow \exists! \rho \\ & & B \end{array}$$

Shilov boundaries

There is a useful description of C^*-envelopes using Shilov ideals.

Definition

Let $\kappa : E \to A$ be a C^*-extension of an operator system. A boundary ideal is given by a closed 2-sided ideal $I \subseteq A$ such that the quotient map $q : A \to A/I$ is completely isometric on $\kappa(E) \subseteq A$.

The Shilov boundary ideal is the largest of such boundary ideals.

Proposition

Let $\kappa : E \to A$ be a C^*-extension. Then there exists a Shilov boundary ideal J and $C^*_{env}(E) \cong A/J$.

As an example consider the operator system of continuous harmonic functions $C_{\mathrm{harm}}(\overline{\mathbb{D}})$ on the closed disc. Then by the maximum modulus principle the Shilov boundary is S^1. Accordingly, its C^*-envelope is $C(S^1)$.

How far is an operator system from a C^*-algebra?

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E.

Definition

The propagation number $\text{prop}(E)$ *of* E *is defined as the smallest integer* n *such that* $(\imath_E(E))^{\circ n} \subseteq C^*_{env}(E)$ *is a* C^**-algebra.*

Returning to the harmonic functions in the disk we have $\text{prop}(C_{\text{harm}}(\overline{\mathbb{D}})) = 1$.

Proposition (Connes-vS, 2020; Pawlowska, 2024)

The propagation number is invariant under complete order isomorphisms, as well as under stable=Morita equivalence [EKT, 2019]:

$$prop(E) = prop(E \otimes_{min} \mathcal{K})$$

More generally [Koot, 2021], we have

$$prop(E \otimes_{\text{min}} F) = \max\{prop(E), prop(F)\}$$

Example: spectral truncation of the circle [Connes-vS, 2020]

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\operatorname{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

- States are defined as unital positive linear functionals.

We have: $C^*_{\text{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\text{prop}(C(S^1)^{(n)}) = 2$ (for any n).

Fourier truncations: the Fejér–Riesz operator system

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.
- The C^*-envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z}) \cong C(S^1)$; propagation number ∞

Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. *The extreme rays in* $C(S^1)^{(n)}_+$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n / S_n$.

Curiosities on Toeplitz matrices

Theorem (Carathéodory)

Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ iff $T = V \Delta V^*$ with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \qquad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \ldots, d_n \geq 0$ and $\lambda_1, \ldots, \lambda_n \in S^1$.

Operator systems, groupoids and bonds (aka a positivity domain)

Definition (Connes-vS, 2021)

A bond is a triple (G, ν, Ω) *consisting of a locally compact groupoid* G*, a Haar system* $\nu = \{\nu_x\}$ *and an open symmetric subset* $\Omega \subseteq G$ *containing the units* $G^{(0)}$*.*

Proposition

Let (Ω, G, ν) *be a bond. The closure of the subspace* $C_c(\Omega) \subseteq C_c(G)$ *in the* C^**-algebra* $C^*(G)$ *is a (possibly non-unital) operator system.*

Example

1. *Consider* Ω *in a l.c. Lie group* $G \rightsquigarrow$ *Fourier truncations (á la Rieffel) in* $C^*(G)$
2. *Consider* $\Omega_n = (-n, n) \subset \mathbb{Z} \rightsquigarrow$ *Fejér–Riesz system* $C^*(\mathbb{Z})_{(n)} \cong (C(S^1)^{(n)})^d$.
3. *Consider* $\Omega_n = (-n, n) \subseteq C_m$ *(so modulo* m*). The operator system consists of banded* $m \times m$ *circulant matrices of band width* n*.*

Tolerance relations on finite sets [Gielen–vS, 2022]

Let X be a finite set and $\mathcal{R} \subseteq X \times X$ a symmetric reflexive relation on X

1. The C^*-envelope of $E(\mathcal{R})$ is $\mathcal{K}(\ell^2(X)) \cong M_{|X|}(\mathbb{C})$ and $\mathrm{prop}(E(\mathcal{R})) = \mathrm{diam}(\mathcal{R})$.
2. If $\mathcal{R}$ is a chordal graph, then $E(\mathcal{R})^d \cong E(\mathcal{R})$ as a vector space, but with order structure given by being partially positive.
3. Concrete realization of $E(\mathcal{R})^d$ in terms of cliques C in $\mathcal{R}$:

$$\Phi : E(\mathcal{R})^d \to \bigoplus_{C \in \mathcal{C}} M_{|C|}(\mathbb{C}); \qquad (x_{ij}) \mapsto ((x_{ij})_{i,j \in C})_{C \in \mathcal{C}}$$

4. the pure states of $E(\mathcal{R})$ are given by vector states $|v\rangle\langle v|$ for which the support of $v \in \ell^2(X)$ is $\mathcal{R}$-connected.

Example

The operator systems of $p \times p$ band matrices with band width N.
The dual operator system consists of partially defined band matrices.

Operator systems associated to tolerance relations

- A metric space (X, d) with the relation

$$\boxed{\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}}$$

- If (X, μ) is a measure space and $\mathcal{R}_\epsilon \subseteq X \times X$ an open subset we obtain the operator system $E(\mathcal{R}_\epsilon)$ as the closure of integral operators with support in $\mathcal{R}_\epsilon$. Note that $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$

Proposition

Let X be a complete, locally compact path metric measure space with a measure of full support. Then $C^*_{env}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$ and

$$\mathrm{prop}(E(\mathcal{R}_\epsilon)) = \lceil diam(X)/\epsilon \rceil$$

The pure states of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is ϵ-connected.

K-theory for operator systems

[arXiv:2409.02773]

A key invariant of C^*-algebras is K-theory. Is there an analogue for operator systems?

- Need notion of projection (*cf.* Araiza–Russell) or invertible selfadjoint elements
- It should capture the spectral localizer of Loring, Schulz-Baldes, and others
- It should be invariant under Morita equivalence [EKT]

Definition

A hermitian form *x in a unital operator system E is a selfadjoint element $x \in M_n(E)$ which is* non-degenerate *in the sense that there exists $g > 0$ such that for all pure and maximal ucp maps $\phi : E \to B(\mathcal{H})$ we have*

$$|\phi^{(n)}(x)| \geq g \cdot \mathrm{id}_{\mathcal{H}}^{\oplus n}$$

In other words, x should have a gap g in each boundary representation

We will write $H(E, n)$ for all hermitian forms in $M_n(E)$.

Proposition

An element $x \in M_n(E)$ is non-degenerate if and only if $\imath_E^{(n)}(x)$ is an invertible element in the C^*-envelope $C^*_{env}(E)$.

This is a consequence of the realization of the C^*-envelope in [Davidson–Kennedy]

Examples:

1. Hermitian forms (à la Witt) on a fgp right module pA^n over a C^*-algebra A: described by invertible elements $x = h + (1 - p) \in M_n(A)$ with $h \in pM_n(A)p$.
2. Projections p in operator systems à la Araiza–Russell are precisely projections in the C^*-envelope: $x = e - 2p$ is a hermitian form.
3. Similarly, ϵ-projections in quantitative K-theory define hermitian forms.
4. Spectral compressions of projections in C^*-algebra: $x = PYP$ with $Y = 1 - 2p$ provided $\|[P, p]\|$ sufficiently small.

The invariants and K-theory

$$\boxed{\mathcal{V}(E,n) = H(E,n)/_{\sim_n}}$$

Example:

$$\mathcal{V}(\mathbb{C},n) \cong \{-n, -n+2, \ldots, n\}$$

and with the map $\imath_{nm}([x]) = x \oplus e_{m-n}$ we have

$$\begin{array}{ccc} \mathcal{V}(\mathbb{C},n) & \xrightarrow{\ \imath_{nm}\ } & \mathcal{V}(\mathbb{C},m) \\ & {\scriptstyle \rho_n} \searrow \quad \swarrow {\scriptstyle \rho_m} & \\ & \mathbb{Z} & \end{array}$$

In general, we consider

$$\mathcal{V}(E) = \varinjlim \mathcal{V}(E,n)$$

and $K_0(E)$ is the corresponding Grothendieck group (with identity $[e]$ and addition $'\oplus'$)

Properties of K_0

- For C^*-algebras we obtain usual K-theory via the map $[x] \mapsto [p = \frac{1}{2}(1 - x|x|^{-1})]$.
- Stability: we define a map $\imath_n : M_n(E) \to M_n(M_2(E))$ by

$$\imath_n(x) = \left(\begin{array}{cc|cc|c|cc} x_{11} & 0 & x_{12} & 0 & \cdots & x_{1n} & 0 \\ 0 & e & 0 & 0 & & 0 & 0 \\ \hline x_{21} & 0 & x_{22} & 0 & \cdots & \multicolumn{2}{c}{\vdots} \\ 0 & 0 & 0 & e & & & \\ \hline \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c}{\vdots} \\ \hline x_{n1} & 0 & \multicolumn{2}{c|}{\cdots} & \cdots & x_{nn} & 0 \\ 0 & 0 & & & & 0 & e \end{array}\right)$$

so that $\imath_n(x) \sim x$ (Whitehead). This allows to show $K_0(E) \cong K_0(M_2(E))$.

Summary

- Noncommutative geometry: metric aspect
- Operator systems: from (spectral) truncations and tolerance relations
- Duality of operator systems: state spaces
- New invariants: propagation number, K-theory
  - Higher K-group invariants [arXiv:2411.02981]:

    $$\mathcal{V}_1^\delta(E,n) = \left\{ x \in M_n(E) : \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \text{ has spectral gap } \delta \right\} /_{\sim_n}$$

    and, more generally, $\mathcal{V}_p^\delta(E,n) := H^\delta(E \otimes \mathbb{C}l_p^{(1)}, n)/_{\sim_n}$.
  - Formal periodicity: $K_{2m}(E) = K_0(E)$ and $K_{2m+1}(E) = K_1(E)$.