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Article

Convergence of Spectral Truncations for Compact Metric Groups

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We study Gromov–Hausdorff convergence of state spaces for truncations of a compact metric group G in momentum space. We work in the context of order-unit spaces and consider orthogonal projections P_Λ in $L^2(G)$ corresponding to finite subsets of irreducible representations $\Lambda \subseteq \widehat{G}$ appearing in the Peter–Weyl decomposition of G. We then prove that the sequence of truncated state spaces $\{S(P_\Lambda C(G) P_\Lambda)\}_\Lambda$ Gromov–Hausdorff converges to the original state space $S(C(G))$, when these are equipped with a metric associated to a Lip-norm which in turn is induced by the action of G.

1 Introduction

This paper deals with the question of how one can approximate the geometric structure of a compact metric group G by finite-dimensional—so-called spectral truncations. We work in the context of quantum metric spaces as introduced by Rieffel [23] (and further developed in [12, 14, 15]) but employ the classical notion of Gromov–Hausdorff convergence for compact metric spaces. The spectral truncations we consider are defined by orthogonal projections onto finite sums of irreducible representations that appear in the Peter–Weyl decomposition of $L^2(G)$. The later yield so-called order-unit spaces, a notion that allows to speak of state spaces. When the order-unit spaces are equipped with a suitable Lip-norm, there is a metric on their state spaces that metrizes the weak* topology. It is in this context that the question concerning Gromov–Hausdorff convergence can be addressed, and, in the present paper, resolved in the case of compact metric groups equipped with a bi-invariant metric.

This work is closely related to some of our previous papers on Gromov–Hausdorff convergence for spectral truncations of Dirac operators on the circle [9, 26] and tori [4, 17]. However, here we treat the case of general compact metric groups equipped with a Lip-norm that is naturally induced by the group action of G (as in [22, 24]). We leave the more involved spectral truncations associated to a Laplace or Dirac-type operator on G—and the corresponding induced Lip-norms—for future research. Also, the main difference with these previous attempts is that we no longer use specifically the formal adjoint of $C(G) \mapsto P_\Lambda C(G) P_\Lambda$ for the reverse map. This freedom enables us to prove a more general convergence result. In fact, we prove that the truncated state space $S(P_\Lambda C(G) P_\Lambda)$ Gromov–Hausdorff converges to the original state space $S(C(G))$, where the index Λ is a finite subset of the dual $\widehat{G}$ (the set of equivalence classes of irreducible representations) of G. The net of truncations is parametrized by the directed set of finite subsets of $\widehat{G}$.

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Our result is also closely related to the recent work [25], which considers discrete groups with polynomial growth. The author uses quantum Gromov–Hausdorff distance instead of Gromov–Hausdorff distance as we do here, but in fact our present results extend *mutatis mutandis* for the former notion of convergence as well. We should note however that the two notions of distance do not coincide in general [10].

Our results mean a great step forward in the study of spectral approximation of geometric spaces, as initiated in [6, 7]. Subsequent results on the quantum Gromov–Hausdorff convergence for quantum groups and/or homogeneous spaces are now of course waiting to be explored. In this regard, an adaptation of our approach to the framework of compact quantum groups have already been carried out in [16]. Finally, let us point out that the first examples of a related but different kind of truncations—so-called Fourier truncations—have been analyzed in [1, 2, 21].

2 Preliminaries

2.1 Order-unit spaces

We start by recalling some basic notions on order-unit spaces, referring to [3, Chapter II] for more details.

Definition 1. An *order-unit space* is a real ordered vector space A together with an archimedean order unit, that is, an element $e \in A$ that satisfies the following:

(1) for each $a \in A$ there is an $r \in \mathbb{R}$ such that $a \leq re$; and
(2) if $a \in A$ and if $a \leq re$ for all $r \in \mathbb{R}_+$ then $a \leq 0$.

A linear map $\phi : A \to B$ between two order-unit spaces is said to be an *order-unit morphism* if it preserves the unit and the order.

Let A be an order-unit space and let $\mathbb{R}$ be equipped with its natural structure of order-unit space. There exists a unique order-unit morphism $\mathbb{R} \to A$ and it is injective. We always consider $\mathbb{R}$ as embedded in A through that morphism. There is also a natural order-unit norm $\|\cdot\|$ on A defined by:

$$\|a\| = \inf\{t > 0 : -t \leq a \leq t\}, \qquad (\forall a \in A).$$

Note that for $\phi : A \to B$ an order-unit morphism, we have:

$$\|\phi(a)\| \leq \|a\|, \qquad (\forall a \in A).$$

Definition 2 (State space). A *state* on an order-unit space A is an order-unit morphism $\sigma : A \to \mathbb{R}$. We denote by $S(A)$ the set of all states on A.

The state space is compact for the weak* topology by a direct application of the Banach–Alaoglu Theorem.

2.2 Lip-norms and Gromov–Hausdorff convergence of state spaces

To introduce a notion of distance for state spaces of order-unit spaces, we follow [23, Section 2] and use Lipschitz semi-norms:

Definition 3 (Lip-norm). Let A be an order-unit space. A *Lip-norm* on A is a seminorm L on A such that:

(1) for $a \in A$ we have $L(a) = 0$ iff $a \in \mathbb{R}$; and
(2) the formula

$$d_L(\sigma, \tau) = \sup\{|\sigma(a) - \tau(a)| : a \in A, L(a) \leq 1\}$$

defines a metric on $S(A)$ which induces the weak* topology.

A convenient way to check the second condition is given by a combination of [22, Proposition 1.6, Theorem 1.9]. Explicitly, assuming that (1) is true, condition (2) holds iff:

(i) L (considered as a norm on $A/\mathbb{R}$) is dominated by the quotient norm on $A/\mathbb{R}$ relative to $\|\cdot\|$; and
(ii) the set $\mathcal{B}_1 := \{a \in A : L(a) \leq 1, \|a\| \leq 1\}$ is totally bounded in A for $\|\cdot\|$.

We point out that the idea of defining metrics on states spaces out of certain seminorms first appeared in [5].

Now, we can define a refinement of the category of order-unit spaces taking into account this metric aspect:

Definition 4. A *quantum metric space* is a pair (A, L) with A an order-unit space and L a Lip-norm on A.
A *morphism* $(A, L_A) \to (B, L_B)$ between two quantum metric spaces is by definition an order-unit morphism $\phi : A \to B$ such that

$$L_B(\phi(a)) \leq L_A(a); \qquad (\forall a \in A).$$

This terminology is justified by the fact that if (A, L) is a quantum metric space, then $(S(A), d_L)$ is a compact metric space (since d_L metrizes the weak* topology). As a matter of fact, we may consider S as a contravariant functor from the category of quantum metric spaces to that of compact metric spaces, and write accordingly $S(A, L) = (S(A), d_L)$ as the state space of (A, L). Indeed, a morphism of quantum metric spaces $\phi : (A, L_A) \to (B, L_B)$ induces by pullpack a continuous function $\phi^* : (S(B), d_{L_B}) \to (S(A), d_{L_A})$ such that $d_{L_A}(\phi^*\sigma, \phi^*\tau) \leq d_{L_B}(\sigma, \tau)$ for all states σ, τ on B.

Example 5. Given a compact metric space (X, d) we may speak about Lipschitz functions as functions $f : X \to \mathbb{R}$ for which there exists a constant $k > 0$ such that

$$|f(x) - f(y)| \leq k\, d(x, y); \qquad (\forall x, y \in X).$$

In that case, the lowest constant k verifying the above property is denoted by $L(f)$.
The algebra $\mathcal{A}$ of all Lipschitz functions on X is a unital sub-space of $C(X)$ (continuous real-valued functions on X) and hence inherits a structure order-unit space. The semi-norm L is actually a Lip-norm on $\mathcal{A}$, so that $(\mathcal{A}, L)$ is a quantum metric space. In this case $S(\mathcal{A}, L) = (\mathcal{M}(X), d)$, the space of Borel probability measures on X equipped with the Kantorovich–Rubinstein metric (cf. [19, 20]).

We end this preliminary section with the following general result on the Gromov–Hausdorff convergence of state spaces [23] (cf. [26, Theorem 5]):

Theorem 6. Let (A, L_A) and (B, L_B) be two quantum metric spaces. Assume that we are given two morphisms of quantum metric spaces $\phi_{AB} : (B, L_B) \to (A, L_A)$ and $\phi_{BA} : (A, L_A) \to (B, L_B)$. Let ϵ be the lowest constant such that for all $a \in A$ and $b \in B$:

$$\|\phi_{AB} \circ \phi_{BA}(a) - a\| \leq \epsilon\, L_A(a)$$

$$\|\phi_{BA} \circ \phi_{AB}(b) - b\| \leq \epsilon\, L_B(b).$$

Then we have the following upper bound for the Gromov–Hausdorff distance between $S(A, L_A)$ and $S(B, L_B)$:

$$d_{GH}(\, S(A, L_A), S(B, L_B)\,) \leq \epsilon.$$

In [26] such a pair of maps (ϕ_{AB}, ϕ_{BA}) was called a C^1-*approximate order isomorphism*.

Remark 7. This result actually extends to yield convergence in the quantum Gromov–Hausdorff distance as well, as already exploited in [25]. We stress that the quantum Gromov–Hausdorff distance and the Gromov–Hausdorff in general may not agree, as witnessed by the interesting paper [10].

3 Truncations of a Compact Metric Group

We first recall the notion of metric group:

Definition 8. A metric d on a group G is said to be *bi-invariant* if:

$$d(gx, gy) = d(x, y) = d(xg, yg); \qquad (\forall g, x, y \in G).$$

A *metric group* is a pair (G, d) with G a group and d a bi-invariant distance on G.

A metric group is in particular a metric space and a topological group. In the following we restrict our attention to *compact metric groups*. For this reason, we use harmonic analysis on compact groups in an essential way. What is needed for our purpose can be found in many monographs, such as [8, 13].

Throughout the rest of this paper, (G, d) is assumed to be a compact metric group. Then there exists a unique Haar probability measure on G. The Hilbert space of L^2 functions with respect to this measure will be denoted simply by $L^2(G)$. The left (resp. right) regular action of G on $L^2(G)$ is denoted by U (resp. V), that is, for all $\psi \in L^2(G)$ and $g, x \in G$ we set

$$(U_g\psi)(x) = \psi(g^{-1}x), \qquad (V_g\psi)(x) = \psi(xg).$$

Recall that the Peter–Weyl Theorem (*cf.* [13, Theorem 1.12]) gives a decomposition of $L^2(G)$ into irreducible unitary representations of G with respect to these left and right actions,

$$L^2(G) = \widehat{\bigoplus}_{\nu\in\widehat{G}} E_\nu \otimes E_\nu^*, \tag{1}$$

where $\widehat{G}$ is the set of equivalent classes of irreducible unitary representations of G. The equality in (1) is intended to stress that in the following we will in fact identify the left- and right-hand side.

We again write $\mathcal{A}$ for the algebra of real-valued Lipschitz functions on (G, d). An element f of $\mathcal{A}$ acts as a bounded self-adjoint operator on $L^2(G)$ acting by pointwise multiplication. In other words, $\mathcal{A}$ can also be considered as a unital sub-space of the order-unit space $\mathcal{B}(L^2(G))_{sa}$ of all bounded self-adjoint operators on $L^2(G)$. Of course, the order-unit structures on $\mathcal{A}$ induced by the inclusion in $C(G)$ and $\mathcal{B}(L^2(G))_{sa}$ are the same.

For all $g \in G$ we define:

$$\begin{array}{cccc} \lambda_g & : & \mathcal{B}(L^2(G))_{sa} & \longrightarrow & \mathcal{B}(L^2(G))_{sa} \\ & & T & \longmapsto & U_g T U_g^* \end{array}$$

and

$$\begin{array}{cccc} \rho_g & : & \mathcal{B}(L^2(G))_{sa} & \longrightarrow & \mathcal{B}(L^2(G))_{sa} \\ & & T & \longmapsto & V_g T V_g^*. \end{array}$$

The maps λ and ρ define two actions of G on $\mathcal{B}(L^2(G))_{sa}$ by order-unit automorphisms. Furthermore, they preserve $\mathcal{A}$; indeed, they induce the maps given by left/right translation of G on itself, that is,

$$\lambda_g(f)(x) = f(g^{-1}x), \qquad \rho_g(f)(x) = f(xg); \qquad (x, g \in G, f \in \mathcal{A}). \tag{2}$$

For $T \in \mathcal{B}(L^2(G))_{sa}$ we may now define three quantities (that may be infinite):

$$\|T\|_\lambda = \sup_{x\neq y} \left\{ \frac{\|\lambda_x(T) - \lambda_y(T)\|}{d(x, y)} \right\}$$

$$\|T\|_\rho = \sup_{x\neq y} \left\{ \frac{\|\rho_x(T) - \rho_y(T)\|}{d(x, y)} \right\}$$

$$\|T\|_{\lambda,\rho} = \max\left(\|T\|_\lambda, \|T\|_\rho \right).$$

Combining these definitions with Equation (2) we conclude that the Lip-norm L on $\mathcal{A}$ (the one from example 5) can be re-expressed as

$$L(f) = \|f\|_\lambda = \|f\|_\rho; \qquad (\forall f \in \mathcal{A}). \tag{3}$$

Note also that L is regular in the sense of [18, 21], that is, it is finite on the linear span of the coordinate elements of all finite-dimensional representations.

3.1 Spectral truncations of compact metric groups

Let us now define the main object of interest to us: the spectral truncations of the metric space (G, d). We would like to start by stressing that we consider truncations induced by projections onto isotypic components of $L^2(G)$ associated to finite subsets of the dual of G, rather than spectral projections of some Dirac operator, as in previous works [4, 6, 9, 17, 26]. However, both approaches are closely related, we refer to Remark 10 and Example 11 below for a discussion on that topic.

More precisely, let $\mathcal{F}$ be the directed set of all finite subsets of $\widehat{G}$. For any $\Lambda \in \mathcal{F}$ we may consider the irreducible representations ν in Λ that appear in $L^2(G)$. In view of Equation (1), we thus consider the Hilbert subspaces

$$L^2(G)_\Lambda := \bigoplus_{\nu \in \Lambda} E_\nu \otimes E_\nu^* \subseteq L^2(G).$$

The orthogonal projections on $L^2(G)$ with image $L^2(G)_\Lambda$ are denoted by P_Λ. Clearly, the algebra $\mathcal{A}$ of Lipschitz functions on G does not restrict to act on $L^2(G)_\Lambda$; instead, we should *compress* $\mathcal{A}$ by P_Λ to act on the Hilbert subspace $L^2(G)_\Lambda$. More generally, we define an order-unit morphism $\tau_\Lambda : \mathcal{B}(L^2(G))_{sa} \to \mathcal{B}(L^2(G)_\Lambda)_{sa}$ by

$$\tau_\Lambda(T) = P_\Lambda T P_\Lambda, \qquad (\forall T \in \mathcal{B}(L^2(G))_{sa}).$$

Here, we identify $\mathcal{B}(L^2(G)_\Lambda)$ with the space of bounded operators T on $L^2(G)$ such that $P_\Lambda T P_\Lambda = T$. Since P_Λ commutes with U_g and V_g for all $g \in G$, the order-unit morphism τ_Λ commutes with the actions λ and ρ. We denote by $\mathcal{A}_\Lambda = P_\Lambda \mathcal{A} P_\Lambda$ the image of $\mathcal{A}$ by τ_Λ. Then both $\mathcal{B}(L^2(G)_\Lambda)_{sa}$ and $\mathcal{A}_\Lambda$ are preserved by λ and ρ.

Proposition 9. Let L_Λ be the restriction of $\|\cdot\|_{\lambda,\rho}$ to $\mathcal{A}_\Lambda$. Then L_Λ is a Lip-norm on $\mathcal{A}_\Lambda$ and $\tau_\Lambda : (\mathcal{A}, L) \to (\mathcal{A}_\Lambda, L_\Lambda)$ is a morphism of quantum metric spaces.

Proof. Let us prove first that $L_\Lambda(T) = 0$ implies $T \in \mathbb{R}$ for all $T \in \mathcal{A}_\Lambda$ (the other implication being obvious). By definition of L_Λ in terms of $\|\cdot\|_{\lambda,\rho}$ we have $L_\Lambda(T) = 0$ iff $\lambda_x(T) = T$ for all $x \in G$. Upon writing $T = \tau_\Lambda(f)$, we then have

$$T = \int \lambda_x(T)\, dx = \tau_\Lambda\left(\int \lambda_x(f)\, dx\right),$$

where the integrals are taken with respect to the Haar probability measure on G. Because of the translation invariance of the latter, the $C(G)$-valued integral $\int \lambda_x(f)\, dx$ is actually a constant function so that $T \in \mathbb{R}$. To show that L_Λ is a Lip-norm on A_Λ, it remains to check properties (i) and (ii) below definition 3. In our case they are automatically verified because A_Λ is finite dimensional.

The fact that $\tau_\Lambda : (\mathcal{A}, L) \to (\mathcal{A}_\Lambda, L_\Lambda)$ is a morphism of quantum metric spaces is straightforward from Equation (3). ∎

Remark 10. As mentioned in the introduction, the truncations that we consider in the present paper are not always associated to the spectrum of some Laplace or Dirac-type operator on G, in contrast to some of our previous works [17, 26] (see also [4, 9]). Supposing that G is a compact *Lie* group (so that Laplace- or Dirac-type operators exist) we would expect these to yield a restriction to a subset $\mathcal{F}'$ of $\mathcal{F}$ corresponding to the eigenspaces of the pertinent operator. The main challenge is then to compare the above Lip-norms L_Λ with the norm of the commutator $\|[D_\Lambda, \cdot]\|$ with the truncated Dirac operator D_Λ.

Example 11. In order to illustrate the previous remark, let us examine the case of the torus, that is $G = \mathbb{R}^r/\mathbb{Z}^r$ and d is the canonical translation invariant metric on it. The Pontryagin dual of G is identified with $\mathbb{Z}^r$. More precisely, for $n \in \mathbb{Z}^r$, we denote by χ_n the character of G defined by $\chi_n(x) = e^{2i\pi n\cdot x}$. Then, the Peter–Weyl decomposition just consists in the fact that $(\chi_n)_{n\in\mathbb{Z}^r}$ is a Hilbert basis of $L^2(G)$. On the other hand, let $S = G \times \Sigma$ be the spinor bundle over G, with Σ the spin representation of Spin(r). The Hilbert space of square integrable spinors, denoted by $L^2(S)$, is identified with $L^2(G) \otimes \Sigma$. The Dirac operator D acts on it as $\sum_{\mu=1}^r \partial_\mu \otimes \gamma^\mu$, where the γ-matrices are skew-adjoint and form a Clifford representation: $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\delta^{\mu\nu}$. For $t > 0$, we write P_t^S for the spectral projection of D relative to the eigenvalues having absolute value less than t. Since $D^2 = -\sum_\mu \partial_\mu^2$, we have $P_t^S = P_{\Lambda_t} \otimes 1_\Sigma$ with $\Lambda_t = \{n \in \mathbb{Z}^r : |n| \leq t\}$, namely

$$P_t^S L^2(S) = \bigoplus_{n\in\Lambda_t} \chi_n \otimes \Sigma.$$

In particular, considering that $\mathcal{A}$ acts on $L^2(S)$ by multiplication, one can identify the order-unit space $P_t^S \mathcal{A} P_t^S$ with $\mathcal{A}_{\Lambda_t} \otimes 1_\Sigma$. Taking into account the connectivity of G, the expression of the Lip-norm L_{Λ_t} simplifies to

$$L_{\Lambda_t}(T) = \sup\{\|x^\mu[\partial_\mu, T]\| : x \in \mathbb{R}^r, |x| = 1\},$$

whereas the Lip-norm coming from the truncated Dirac operator $D_t = P_t D P_t$ is expressed as

$$\|[D_t, T \otimes 1_\Sigma]\| = \|[\partial_\mu, T] \otimes \gamma^\mu\|.$$

Here we used twice Einstein's summation convention. Both Lip-norms are obviously equal for the circle (i.e., $r = 1$), which has already been noted in [6]. This is much less clear for higher dimensional tori although we expect that $L_{\Lambda_t} \leq \|[D_t, \cdot]\|$ in general. In that case, the main result of the present paper would be stronger than [17, Theorem 3.10].

4 Gromov–Hausdorff Convergence of State Spaces

We now establish that the net of compact metric spaces $(S(\mathcal{A}_\Lambda, L_\Lambda))_{\Lambda\in\mathcal{F}}$ converges to $S(\mathcal{A}, L)$ in Gromov–Hausdorff sense. Our strategy consists in constructing morphisms $\upsilon_\Lambda : (\mathcal{A}_\Lambda, L_\Lambda) \to (\mathcal{A}, L)$ and then apply Theorem 6 to the maps υ_Λ and τ_Λ.

Remark 12. Note that for any $\Lambda \in \mathcal{F}$ the order-unit morphism $\tau_\Lambda : \mathcal{A} \to \mathcal{A}_\Lambda$ is surjective. As a consequence, the induced map at the level of state spaces,

$$\begin{array}{cccc} \tau_\Lambda^* : & S(\mathcal{A}_\Lambda) & \longrightarrow & S(\mathcal{A}) \\ & \sigma & \longmapsto & \sigma \circ \tau_\Lambda, \end{array}$$

is injective. Since $\mathcal{A}$ is a dense unital sub-space of $C(G)$, the state space $S(\mathcal{A})$ coincides with $S(C(G))$, which which can be identified with the set of Borel probability measures on G.

Definition 13. Let $\Lambda \in \mathcal{F}$. A Borel probability measure μ on G (considered as an element of $S(\mathcal{A})$) is said to be **liftable** (or that it can be lifted) to a state on $\mathcal{A}_\Lambda$ if $\mu \in \tau_\Lambda^* S(\mathcal{A}_\Lambda)$. In this case the unique antecedent of μ by τ_Λ^* will be denoted by μ_Λ; it thus satisfies $\mu_\Lambda \circ \tau_\Lambda = \mu$.

Remark 14. If μ is a Borel probability measure that is liftable to a state on $\mathcal{A}_{\Lambda_0}$ then for all $\Lambda \in \mathcal{F}$ containing Λ_0, μ can also be lifted to a state on $\mathcal{A}_\Lambda$ and we have

$$\mu_\Lambda = \mu_{\Lambda_0} \circ (\tau_{\Lambda_0|\mathcal{A}_\Lambda}),$$

with $\tau_{\Lambda_0|\mathcal{A}_\Lambda}$ the restriction of τ_Λ to an order-unit morphism $\mathcal{A}_\Lambda \to \mathcal{A}_{\Lambda_0}$.

Proposition 15. Let $\Lambda \in \mathcal{F}$ and μ a Borel probability measure on G which is liftable to a state on $\mathcal{A}_\Lambda$. Let $\upsilon^\mu_\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}$ be the order-unit morphism defined by the following:

$$\upsilon^\mu_\Lambda(T)(g) = \mu_\Lambda(\rho_g(T)), \qquad (\forall T \in \mathcal{A}_\Lambda, \forall g \in G).$$

Then $\upsilon^\mu_\Lambda : (\mathcal{A}_\Lambda, L_\Lambda) \to (\mathcal{A}, L)$ is a morphism of quantum metric spaces.
Moreover, we have

$$d_{GH}(S(\mathcal{A}_\Lambda, L_\Lambda), S(\mathcal{A}, L)) \le \int_G d(e, x) d\mu(x).$$

Proof. The fact that υ^μ_Λ is an order-unit morphism is straightforward. For continuity with respect to the Lip-norm we compute

$$\begin{aligned} L(\upsilon^\mu_\Lambda(T)) &= \sup_{x \neq y} \frac{\left|\upsilon^\mu_\Lambda(T)(x) - \upsilon^\mu_\Lambda(T)(y)\right|}{d(x, y)} \\ &= \sup_{x \neq y} \frac{\left|\mu_\Lambda(\rho_x(T) - \rho_y(T))\right|}{d(x, y)} \le \|T\|_\rho. \end{aligned}$$

This shows that υ^μ_Λ is a morphism of quantum metric spaces.

We establish the second claim by showing that for the pair $(\tau_\Lambda, \upsilon^\mu_\Lambda)$ the constant ϵ in Theorem 6 has the following upper bound:

$$\epsilon \le \int_G d(e, x) d\mu(x).$$

We first consider the approximation property of the map $\upsilon^\mu_\Lambda \circ \tau_\Lambda$. Let $f \in \mathcal{A}$ and let us abbreviate $f_\Lambda \equiv \upsilon^\mu_\Lambda \circ \tau_\Lambda(f)$. We have

$$f_\Lambda(g) = \int_G f(xg) d\mu(x), \qquad (\forall g \in G),$$

or, equivalently, as a $C(G)$-valued integral (indifferently interpreted in the weak sense or in Bochner sense, just as the other integrals in this proof):

$$f_\Lambda = \int_G \lambda_{x^{-1}}(f)\, d\mu(x).$$

Hence, we have

$$\begin{aligned} \|f_\Lambda - f\| &= \left\| \int_G (\lambda_{x^{-1}}(f) - f) d\mu(x) \right\| \le \int_G \|\lambda_{x^{-1}}(f) - f\|\, d\mu(x) \\ &\le \int_G L(f)\, d(x^{-1}, e)\, d\mu(x) \; = \; L(f) \int_G d(e, x)\, d\mu(x). \end{aligned}$$

For the composition $\tau_\Lambda \circ \upsilon^\mu_\Lambda$ consider an arbitrary $T = \tau_\Lambda(f) \in \mathcal{A}_\Lambda$. We then have $\upsilon^\mu_\Lambda(T) = f_\Lambda$ in the notation of the previous paragraph, so that

$$\tau_\Lambda \circ \upsilon^\mu_\Lambda(T) = \int_G \lambda_{x^{-1}}(T)\, d\mu(x),$$

as an operator-valued integral, where we also used that P_Λ commutes with $\lambda_{x^{-1}}$ for all $x \in G$. But then it follows as before that

$$\|\tau_\Lambda \circ \upsilon_\Lambda^\mu(T) - T\| = \left\| \int_G (\lambda_{x^{-1}}(T) - T) d\mu(x) \right\| \leq \int_G \|\lambda_{x^{-1}}(T) - T\| \, d\mu(x)$$
$$\leq \int_G L(T)\, d(x^{-1}, e)\, d\mu(x) \; = \; L(T) \int_G d(e, x)\, d\mu(x).$$

We thus obtain an upper bound for ϵ in Theorem 6 from which the claimed upper bound on the Gromov–Hausdorff distance between $S(\mathcal{A}_\Lambda, L_\Lambda)$ and $S(\mathcal{A}, L)$ directly follows. ∎

In view of the last proposition, the remaining task in proving the Gromov–Hausdorff convergence is to find a liftable μ for which the integral $\int_G d(e,x) d\mu(x)$ tends to 0. We will take μ as close as possible to the Dirac mass δ_e with respect to the weak topology; more precisely, we have the following:

Proposition 16. Let $S_{\mathcal{F}} = \bigcup_{\Lambda \in \mathcal{F}} \tau_\Lambda^* S(\mathcal{A}_\Lambda) \subseteq S(\mathcal{A})$ be the set of all Borel probability measures that can be lifted to a state on some $\mathcal{A}_\Lambda$. Then $S_{\mathcal{F}}$ is a dense subset of $S(\mathcal{A})$ for the weak* topology. In particular, for all $\epsilon > 0$ there exists $\mu \in S_{\mathcal{F}}$ such that

$$\int_G d(e, x) d\mu(x) \leq \epsilon. \tag{4}$$

Proof. It is a standard result that density of the convex set $S_{\mathcal{F}}$ in $S(\mathcal{A})$ will follow from the following equality: (see, for instance, [11, Theorem 4.3.9]):

$$\|f\| = \sup \{|\sigma(f)| : \sigma \in S_{\mathcal{F}}\}, \qquad (\forall f \in \mathcal{A}).$$

Let $L^2(G)_{\mathcal{F}} = \sum_{\Lambda \in \mathcal{F}} L^2(G)_\Lambda$. Since $L^2(G)_{\mathcal{F}}$ is a dense subspace of $L^2(G)$, we have for all $f \in \mathcal{A}$:

$$\|f\| = \sup \{|\langle \psi, f\psi \rangle| : \psi \in L^2(G)_{\mathcal{F}}, \|\psi\| = 1\}.$$

But $\langle \psi, \cdot \psi \rangle \in S_{\mathcal{F}}$ for all $\psi \in \mathcal{F}$ such that $\|\psi\| = 1$, so that we may conclude that $S_{\mathcal{F}}$ is a dense subspace of $S(\mathcal{A})$ for the weak$*$ topology.

The state $\mu \in S_{\mathcal{F}}$ such that Equation (4) holds can then be found as follows. Let δ_e be the state on $\mathcal{A}$ defined for all $f \in \mathcal{A}$ by $\delta_e(f) = f(e)$. The function $\Delta : x \mapsto d(e, x)$ is a Lipschitz function, hence an element of $\mathcal{A}$, so by weak$*$ density of $S_{\mathcal{F}}$ for all $\epsilon > 0$ there exists $\mu \in S_{\mathcal{F}}$ such that

$$\int_G d(e, x)\, d\mu(x) = |\mu(\Delta) - \delta_e(\Delta)| \leq \epsilon,$$

as desired. ∎

Now we can finally state and prove our main theorem:

Theorem 17. The net of compact metric spaces $(S(\mathcal{A}_\Lambda, L_\Lambda))_{\Lambda \in \mathcal{F}}$ converges to $S(\mathcal{A}, L)$ in Gromov–Hausdorff sense.

Proof. Let $\epsilon > 0$ and let $\mu \in S_{\mathcal{F}}$ be a liftable probability measure such as in Proposition 16, so that Equation (4) holds. In particular, there is a $\Lambda_0 \in \mathcal{F}$ such that μ is liftable to a state on $\mathcal{A}_{\Lambda_0}$. Then for all $\Lambda \in \mathcal{F}$ containing Λ_0, the measure μ is liftable to a state on $\mathcal{A}_\Lambda$. Applying Proposition 15 then yields for all $\Lambda \in \mathcal{F}$ such that $\Lambda_0 \subseteq \Lambda$ we have

$$d_{GH}(S(\mathcal{A}_\Lambda, L_\Lambda), S(\mathcal{A}, L)) \leq \epsilon,$$

which completes the proof. ∎

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