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Quadratic Forms, Real Zeros and Echoes of the Spectral Action

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Dedicated to Huzihiro Araki with gratitude and admiration

Abstract: For a real distribution $\mathcal{D}$ on the interval $[0, L]$ with $\widetilde{\mathcal{D}}$ the associated even distribution on the interval $[-L, L]$, we prove that if the associated quadratic form with Schwartz kernel $\widetilde{\mathcal{D}}(x-y)$ defines a lower-bounded selfadjoint operator on $L^2([-\frac{L}{2}, \frac{L}{2}])$, whose lowest spectral value λ is a simple, isolated eigenvalue with even eigenfunction ξ, then all the zeros of the entire function $\widehat{\xi}(z)$, the Fourier transform of ξ, lie on the real line. The proof proceeds in five steps. (1) We give a C^*-algebraic proof of a corollary of Carathéodory–Fejér's 1911 structure theorem for Toeplitz matrices: if $T \in M_n(\mathbb{C})$ is a Hermitian, positive semidefinite Toeplitz matrix of rank $n-1$, and $\xi \in \ker T$, then the polynomial $P(z) = \sum \xi_j z^j$ has all its zeros on the unit circle. (2) We formulate and prove a continuous analogue of this result, replacing the Toeplitz matrix with a convolution operator with continuous kernel $h(x-y)$, and the polynomial $P(z)$ with the Fourier transform of the eigenfunction corresponding to the largest eigenvalue. (3) We analyze finite-dimensional truncations of the quadratic forms defined by real, even distributions $\mathcal{D}$ on $[-L, L]$, and observe that the resulting matrices exhibit a structure previously encountered in perturbative expansions of the spectral action. (4) We establish an analogue of Carathéodory–Fejér's corollary for matrices of this specific structure, thereby extending the zero localization result beyond the classical Toeplitz setting. (5) Finally, we apply a classical theorem of Hurwitz concerning the zeros of uniform limits of holomorphic functions to deduce the general result stated above.

Contents

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1. Introduction

The Carathéodory-Fejér theorem from 1911 [5] (see also [4, Theorem 1.3.6]) describes the structure of Hermitian, positive semidefinite Toeplitz matrices as follows. Let

$$T = \begin{bmatrix} c_0 & \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_n \\ c_1 & c_0 & \bar{c}_1 & \cdots & \bar{c}_{n-1} \\ c_2 & c_1 & c_0 & \cdots & \bar{c}_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix}.$$

Then, if T is positive semidefinite of rank r, there exist distinct points $z_1, \ldots, z_r \in \mathbb{T} \subset \mathbb{C}$ (the unit circle), and positive weights $\alpha_1, \ldots, \alpha_r > 0$, such that

$$T = VDV^*,$$

where $V \in \mathbb{C}^{(n+1)\times r}$ is the Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_r \\ z_1^2 & z_2^2 & \cdots & z_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^n & z_2^n & \cdots & z_r^n \end{bmatrix},$$

and $D = \operatorname{diag}(\alpha_1, \ldots, \alpha_r)$ is a diagonal matrix with positive real entries.

In our work on operator systems [10], we showed how the above theorem can be derived from the duality theory of operator systems. This factorization also plays a central role in the proof of truncated Weil positivity in [7].

A direct corollary of the Carathéodory-Fejér theorem is the following:

Corollary 1.1. *Let $T \in M_{n+1}(\mathbb{C})$ be a Hermitian, positive semidefinite Toeplitz matrix of rank n, and let $\xi \in \ker T$. Then all the zeros of the polynomial*

$$P(z) := \sum_{j=0}^{n} \xi_j z^j$$

lie on the unit circle.

This corollary exhibits a striking number-theoretic flavor, resonating with the analogue of the Riemann Hypothesis for function fields; see [12] for a further discussion of this connection. In number theory, Toeplitz matrices of this kind naturally arise, and Corollary 1.1 applies to show that the zeros of the polynomial $P(z)$, associated to an eigenvector for the smallest eigenvalue of such a matrix, all lie on the unit circle. The key difficulty in this context, then, becomes the verification that zero is indeed the (simple) minimal eigenvalue of T.

In the present paper, we investigate a distributional analogue of Corollary 1.1, motivated by its potential relevance to the Riemann Hypothesis itself. Our approach proceeds through several stages:

(1) We give a C^*-algebraic proof of Corollary 1.1.
(2) We formulate and prove a continuous analogue of Corollary 1.1, in which the Toeplitz matrix is replaced by a convolution operator with continuous kernel $h(x-y)$, and the polynomial $P(z)$ by the Fourier transform of the eigenfunction corresponding to the largest eigenvalue.
(3) We analyze the finite truncations of quadratic forms defined by real even distributions $\mathcal{D}$ supported on $[-L, L]$, and observe that the resulting matrices exhibit a structure previously encountered in perturbative expansions of the spectral action.
(4) We prove an analogue of Corollary 1.1 for matrices of this special type.
(5) Finally, using a classical result of Hurwitz on the zeros of uniform limits of holomorphic functions, we deduce the following general theorem:

Theorem 1.2. *Let $L > 0$, $\mathcal{D}$ be a real distribution on the interval $[0, L]$ and $\widetilde{\mathcal{D}}$ the associated even distribution on $[-L, L]$. Assume that the quadratic form with Schwartz kernel $\widetilde{\mathcal{D}}(x-y)$ defines a lower-bounded selfadjoint operator A on $L^2([-\frac{L}{2}, \frac{L}{2}])$, and that the minimum of its spectrum is a simple, isolated eigenvalue λ, with even eigenfunction ξ. Then all the zeros of the entire function $\widehat{\xi}(z)$, $z \in \mathbb{C}$, Fourier transform of ξ lie on the real line.*

We refer to Theorem 6.1 for the precise formulation, the above formulation is slightly unprecise as shown in Remark 4.3. In the course of the proof, we encounter a number of illustrative examples and special cases. A detailed matrix-based verification of the theorem is given in an appendix.

2. Toeplitz case

Recall the operator system $C^*(\mathbb{Z})_{(n+1)} \subseteq C^*(\mathbb{Z})$ given by Fourier truncations on the interval $[-n, n] \subset \mathbb{Z}$ from [10] It is the dual operator system of the operator system of Toeplitz matrices, which in particular allows to associate a positive linear form $\mathcal{L}_T$ to any positive Toeplitz matrix T:

$$\mathcal{L}_T : C^*(\mathbb{Z})_{(n+1)} \to \mathbb{C}; \qquad f \mapsto \sum_{k=-n}^{n} f_k c_k.$$

Note that a positive real Toeplitz matrix T of size $n+1$ can be written in the following form:

$$T = \begin{pmatrix} c_0 & c_1 & \dots & c_n \\ c_1 & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ c_n & \dots & c_1 & c_0 \end{pmatrix}; \qquad (c_k \in \mathbb{R}).$$

The starting point for our proof of Corollary 1.1 is the following purely $*$-algebraic result:

Proposition 2.1. *Let* $T = (c_k)$ *be a positive* $(n+1)$*-dimensional real Toeplitz matrix of rank* n *and let* $(a_j)_{j=0}^n$ *be a real vector in* $\ker T$.

(1) The ideal J *of* $A = \mathbb{C}[X, X^{-1}]$ *generated by* $P = \sum_0^n a_j X^j$ *is stable under the canonical involution,* $(aX^n)^* := \overline{a}X^{-n}$*, of* $\mathbb{C}[X, X^{-1}]$*.*
(2) The monomials X^j $(j = 0, \ldots, n-1)$ *form a basis of* A/J*.*
(3) There exists a unique linear form ϕ *on the quotient* A/J *such that*

$$\phi(X^j) = c_j, \forall j \in \{0, \ldots, n-1\} \tag{1}$$

(4) The linear form ϕ *is positive on the* $*$*-algebra* A/J*.*

Proof. (1) We have that $J^* = J$ since $P(X)$ is either palendromic, or anti-palendromic, *i.e.* $a_{n-j} = \pm a_j$ for all $j = 0, \ldots, n$. Indeed, because of the structure of T as a Toeplitz matrix, it follows that if $(a_j)_j$ is in the kernel of T, so is $(a_{n-j})_j$. Since furthermore this kernel is one-dimensional, it follows that $a_{n-j} = \lambda a_j$, hence $a_j = \lambda^2 a_j$, which implies $\lambda = \pm 1$. But then

$$P^*(X) = \sum_{j=0}^n a_j X^{-j} = \pm X^{-n} P(X)$$

as claimed.

(2) Since T is of rank n one has $a_0 \neq 0$ (see [12, Lemma 33] or show directly that if $a_0 = 0$ then $(a_{j+1})_{j=0}^n$ with $a_{n+1} \equiv 0$ is also in $\ker T$). Thus X is invertible in $\mathbb{C}[X]/J'$ where $J' = P\mathbb{C}[X]$ since modulo P one has $a_0 + XQ = 0$ for $Q = (P - a_0)/X$. The ring $A = \mathbb{C}[X, X^{-1}]$ is the localisation of $\mathbb{C}[X]$ at X and localisation commutes with quotients [2, Proposition 3.3 and Corollary 3.4], since X is not a zero divisor modulo J'. Since X is invertible in $\mathbb{C}[X]/J'$ localization at X does not change $\mathbb{C}[X]/J'$ and one thus gets the equality $A/J = \mathbb{C}[X]/J'$ and the claim follows.

(3) Follows from (2).

(4) Let $\phi : A \to \mathbb{C}$ be the unique linear form which vanishes on the ideal J and fulfills (1). In order to show that ϕ is positive, we first show that $\phi(X^{-j}) = c_j$ for all $j = 1, \ldots, n-1$. We have

$$\begin{aligned} \phi(X^{-1}P) = 0 \implies a_0\phi(X^{-1}) + \sum_{j=1}^n a_j \phi(X^{j-1}) = 0 \\ \implies a_0\phi(X^{-1}) + \sum_{j=1}^n a_j c_{j-1} = 0. \end{aligned}$$

Since $(a_j) \in \ker T$ we have in terms of the second row of T that

$$\sum_{j=0}^n c_{j-1} a_j = 0$$

and hence $\phi(X^{-1}) = c_{-1} = c_1$. This argument can be repeated by considering subsequent rows in T to obtain by induction that $\phi(X^{-k}) = c_{-k} = c_k$. Note that from the first row of T it also follows that $\phi(X^n) = c_n$. We conclude that $\phi(X^j) = c_{|j|}$ for all $j = -n, \ldots, n$.

To show that ϕ is positive, *i.e.* that $\phi(f^* * f) \geq 0$ for all $f \in A$, note that the value of $\phi(f^* * g)$ only depends on the classes of f and g in A/J. Thus one can take $f = \sum_0^{n-1} f_j X^j$, one then gets

$$\phi(f^* * f) = \sum f_j \overline{f_k} \phi(X^{j-k}) = \sum f_j \overline{f_k} c_{|j-k|} = \langle f \mid Tf \rangle \geq 0$$

which shows (4).

□

Proposition 2.2. *Let $T = (c_k)$ be a positive $(n+1)$-dimensional Toeplitz matrix of rank n. If $(a_j)_{j=0}^n$ is a vector in* $\ker T$ *then the polynomial $P(z) = \sum_j a_j z^j$ has all zeros on unit circle in $\mathbb{C}$.*

Proof. The positive linear form ϕ on A vanishing on J defines a positive linear form on the envelopping C^*-algebra $C^*(\mathbb{Z})$ of the involutive algebra A, *i.e.* a positive measure on the Pontrjagin dual $U(1)$ of $\mathbb{Z}$. This measure is supported by the n eigenvalues of the unitary $\pi(X)$ associated to X in the GNS representation π of (A, ϕ) which is of dimension n by construction. Since $P \in J \subset \ker \pi$, these eigenvalues correspond to the roots of the generator P of J which are hence all of modulus one. □

Remark 2.3. In general, when the kernel of T is more than one-dimensional —in other words, when the extreme eigenvalue is not simple— it follows from the Carathéodory-Fejér decomposition that there is an equivalence between being in the kernel and the vanishing of the corresponding polynomial on the complex numbers of modulus one that appear in this decomposition. So if it happens that the number of these complex numbers is strictly less than n, then this condition will be fulfilled by polynomials which will have these particular complex numbers as zeros, but which otherwise can have arbitrary other zeros. This means that it is not true in general if the eigenvalue is not simple, that the theorem holds. The correct formulation of the theorem is that if you take the intersection of the zeros of the various eigenfunctions, then they are all on the unit circle. This is reminiscent to the notion of the radical of a quadratic form.

3. The Continuous Kernel Case

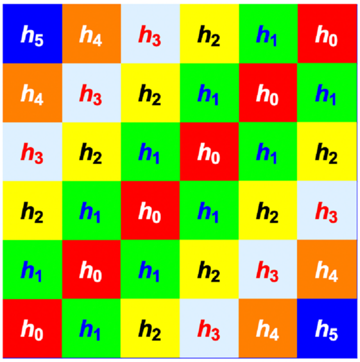
In this section we shall extend the Toeplitz case to the continuous case.

Theorem 3.1. *Let $h \in C([-L, L])$ be an even real continuous function. Let K be the operator on $L^2([0, L])$ given by*

$$(Kf)(x) = \int h(x-y) f(y) dy. \tag{2}$$

Then K is a compact selfadjoint operator. Assume that its eigenvalue of largest modulus is simple and let $\xi \in L^2([0, L])$ be its eigenvector. If we extend ξ to an element of $L^2(\mathbb{R})$ to be zero outside $[0, L]$ then all the zeros of the entire function $\widehat{\xi}$ belong to $\mathbb{R} \subset \mathbb{C}$.

Proof. The operator K of (2) is of Hilbert-Schmidt class since $h(x-y)$ is square integrable. We first approximate K by finite rank operators as follows (Fig. 1). Since the function h is real continuous and even on $[-L, L]$ we can find, given $\epsilon > 0$ an $\alpha > 0$



$$\begin{pmatrix} h_0 & h_1 & h_2 & h_3 & h_4 & h_5 \\ h_1 & h_0 & h_1 & h_2 & h_3 & h_4 \\ h_2 & h_1 & h_0 & h_1 & h_2 & h_3 \\ h_3 & h_2 & h_1 & h_0 & h_1 & h_2 \\ h_4 & h_3 & h_2 & h_1 & h_0 & h_1 \\ h_5 & h_4 & h_3 & h_2 & h_1 & h_0 \end{pmatrix}$$

Fig. 1. The approximation of $h(|x-y|)$ and the reflected Toeplitz matrix (by symmetry with respect to the x axis)

and scalars $h_j \in \mathbb{R}$ for $j = 0, \dots, L/\alpha = N \in \mathbb{N}$, such that, with $j(x)$ denoting the integer part of x/α, one has

$$|h(x-y) - h_{|j(x)-j(y)|}| \leq \epsilon, \qquad \forall x, y \in [0, L].$$

This follows provided one chooses $\alpha > 0$ and the scalars $h_j \in \mathbb{R}$ such that $\forall x \in [0, L]$

$$|h(x) - h_j| \leq \epsilon, \qquad \forall j \mid |j - j(x)| \leq 1,$$

as one gets by comparing $j(|x-y|)$ with $|j(x) - j(y)|$.

Let then χ_j be the characteristic function of the interval $I_j := \{x \mid j(x) = j\}$ and T the Toeplitz matrix $T_{n,m} := h_{|n-m|}$, one thus obtains the inequality

$$|h(x-y) - \sum T_{n,m} \chi_n(x) \chi_m(y)| \leq \epsilon, \qquad \forall x, y \in [0, L].$$

It follows from the Hilbert-Schmidt control of the norm that one obtains in this manner a sequence R_n of finite rank operators converging in norm to K, each of the form, with T a real symmetric Toeplitz matrix and χ_j characteristic functions of intervals,

$$R_n = \sum T_{i,j} |\chi_i\rangle\langle\chi_j|, \quad \|K - R_n\| \to 0.$$

Let then ξ be an eigenvector of norm 1 for the eigenvalue λ of K of largest modulus; without loss of generality we assume that $\lambda > 0$. The spectral projection P obtained

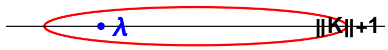


Fig. 2. The contour of integration for the definition of the projection P in Eq. (3)

as a Cauchy integral of the resolvent of K along the contour C (see Fig. 2), isolating λ from the rest of Spec$K$,

$$P = \frac{1}{2\pi i}\int_C (z-K)^{-1}dz \tag{3}$$

is of rank one and fulfills $P\xi = \xi$. Let P_n be defined by the same formula using R_n in place of K. This makes sense for n large enough since $\|K - R_n\| \to 0$ when $n \to \infty$. One has $P_n \to P$ in norm and thus P_n is of rank one for n large enough. Let $\xi_n = P_n\xi$. One has $\xi_n \to \xi$ in norm. Let us show that for n large enough, ξ_n is an eigenvector of R_n for the largest eigenvalue.

By construction, for n large enough, it is a non-zero eigenvector of R_n and is the only one for the interval between the lower part of C and its highest part. Since we have chosen this highest part to be $\|K\|+1$, it follows that, for n large enough, it is the highest eigenvalue of R_n. We can thus assert that ξ is the norm limit of the sequence ξ_n where, for each n, ξ_n is of the form,

$$\xi_n = \sum a_j\chi_j, \quad \sum a_j z^j = 0 \Rightarrow |z| = 1$$

and where the χ_j are the characteristic functions of $N = L/\alpha$ intervals $I_j = I_0 + j\alpha$, $0 \le j < N$ forming a partition of $[0, L)$. We now compute the Fourier transform of ξ_n. The Fourier transform of χ_0 is

$$\widehat{\chi_0}(s) = \int_0^\alpha \exp(-isx)\,dx = \frac{i\left(-1+e^{-i\alpha s}\right)}{s}.$$

One has $\chi_j(x) = \chi_0(x - j\alpha)$, and this gives $\widehat{\chi_j}(s) = \exp(-is\alpha j)\widehat{\chi_0}(s)$ so that we obtain

$$\widehat{\xi_n}(s) = P(\exp(-is\alpha))\widehat{\chi_0}(s), \quad P(z) := \sum a_j z^j$$

Thus the zeros of $\widehat{\xi_n}$ are the union of the set $\{\frac{2n\pi}{\alpha} \mid n \in \mathbb{Z}, n \neq 0\}$ of zeros of $\widehat{\chi_0}$ with the set of complex numbers s such that $\exp(-is\alpha)$ is one of the roots of $P(z)$. But we know (from the Toeplitz case, Proposition 2.2) that all these roots are of modulus 1. For each root z_k of $P(z) = 0$, let $s_k \in \mathbb{R}$ be such that $\exp(-is_k\alpha) = z_k$, the set Z of zeros of $\widehat{\xi_n}$ is then

$$Z = \{\frac{2n\pi}{\alpha} \mid n \in \mathbb{Z}, n \neq 0\} \cup_k \{s_k + \frac{2n\pi}{\alpha} \mid n \in \mathbb{Z}\} \subset \mathbb{R}.$$

Now since the sequence ξ_n converges to ξ in $L^2[0, L]$, we find for the entire functions $\widehat{\xi_n}(s)$ and $\widehat{\xi}(s)$ that

$$|\widehat{\xi}(s) - \widehat{\xi_n}(s)| \le \|\xi - \xi_n\|_{L^2}\int_0^L e^{2\Im(s)x}.$$

We conclude from this that the sequence $\widehat{\xi_n}(s)$ converges uniformly to $\widehat{\xi}(s)$ on compact subsets of $\mathbb{C}$. Thus the Hurwitz's theorem shows that all the zeros of $\widehat{\xi}(s)$ are real. □

4. Quadratic Form Q Associated to a Distribution

Let $L > 0$. We start with a distribution $\mathcal{D}$ on the interval $[0, L]$, *i.e.* a continuous linear form on $C^\infty([0, L])$ written formally as

$$\mathcal{D}(f) = \int_0^L f(x)\mathcal{D}(x), \ \ \forall f \in C^\infty([0, L])$$

and we use it to define an even distribution $\widetilde{\mathcal{D}}$ on $[-L, L]$ by symmetrization,

$$\widetilde{\mathcal{D}}(f) := \mathcal{D}(f) + \mathcal{D}(\widetilde{f}), \ \ \widetilde{f}(x) := f(-x), \ \forall x \in [-L, L]. \tag{4}$$

Note that $\widetilde{\mathcal{D}}(f)$ continues to make sense when f is smooth when restricted to both $[0, L]$ and $[-L, 0]$ but not necessarily smooth at 0. We consider the Hilbert space $\mathcal{H} := L^2([0, L], dx)$ with orthonormal basis

$$U_n(x) := L^{-\frac{1}{2}} \exp(2\pi i n x/L), \ \forall x \in [0, L], \ n \in \mathbb{Z}. \tag{5}$$

These functions are extended to $x \in \mathbb{R}$ so that they vanish for $x \notin [0, L]$. One then uses the involution $f^*(x) := \overline{f(-x)}$ and the convolution

$$(f * g)(y) := \int f(x)g(y - x)dx.$$

We then use $\mathcal{D}$ to get a densely defined hermitian form Q on trigonometric polynomials by

$$\langle f|g\rangle_Q := \widetilde{\mathcal{D}}(f^* * g) = \int_0^L \big((f^* * g)(x) + (f^* * g)(-x)\big)\,\mathcal{D}(x), \tag{6}$$

whose matrix in the orthonormal basis U_n is given by

$$\langle U_m|U_n\rangle_Q := \int_0^L \big((U_m^* * U_n)(y) + (U_m^* * U_n)(-y)\big)\,\mathcal{D}(y)dy. \tag{7}$$

We assume that this defines a lower-bounded self-adjoint quadratic form with spectrum having an isolated eigenvalue at its minimum and are interested in the eigenvector η for this lowest eigenvalue (assumed simple).
For $\mathcal{D} = \delta_0$ the Dirac mass at $y = 0$, one has

$$\begin{aligned}\int_0^L \big((U_m^* * U_n)(y) + (U_m^* * U_n)(-y)\big)\,\mathcal{D}(y)dy &= 2(U_m^* * U_n)(0) = 2\int_0^L U_n(x)\overline{U_m(x)}dx \\ &= 2\langle U_m|U_n\rangle\end{aligned}$$

which shows that by adding to $\mathcal{D}$ a multiple of δ_0 one can assume that the quadratic form Q is positive and the eigenvector η is in the radical of the quadratic form, *i.e.* that one has

$$\langle U_n|\eta\rangle_Q = 0, \ \ \forall n \in \mathbb{Z}.$$

By analogy with the Toeplitz case, one would like to obtain a positive linear form on the quotient of the convolution algebra of functions on $\mathbb{R}$ by the ideal generated by η allowing one to extend the quadratic form Q. This would then yield an analogue of Corollary 1.1 by showing that the zeros of the entire function $\widehat{\eta}$ are all real.

Instead of working directly with this infinite dimensional situation, our strategy is to first compute the matrix (7), observe that it is of a particular form already met in perturbation theory of the spectral action, and prove a matrix form of the reality of zeros of Fourier transforms of lowest eigenvectors. The infinite dimensional result will then follow by approximation using Hurwitz theorem as in the proof of Theorem 3.1.

4.1. Matrix of the quadratic form Q. For $y \in [0, L]$ one has, for $n \neq m$, using $U_m(t) = 0$ for $t < 0$ and $U_n(x) = 0$ for $x > L$,

$$(U_m^* * U_n)(y) = \int U_m^*(y-x)U_n(x)dx = \int \overline{U_m(x-y)}U_n(x)dx$$
$$= \frac{1}{L}\int_y^L \exp(2\pi i m(y-x)/L + 2\pi i n x/L)dx =$$
$$\frac{\exp(2\pi i m y/L)}{L}\int_y^L \exp(2\pi i(n-m)x/L)dx = \frac{\exp(2\pi i m y/L)}{2\pi i(n-m)}\left(\exp(2\pi i(n-m)x/L)\right)_y^L =$$
$$= \frac{\exp(2\pi i m y/L) - \exp(2\pi i n y/L)}{2\pi i(n-m)}.$$

Moreover and still with $y \in [0, L]$ one has

$$(U_m^* * U_n)(-y) = \overline{(U_m^* * U_n)^*(y)} = \overline{(U_n^* * U_m)(y)}.$$

Thus since the formula $\frac{\exp(2\pi i m y/L) - \exp(2\pi i n y/L)}{2\pi i(n-m)}$ is symmetric in n, m, one obtains for $n \neq m$ and $y \in [0, L]$,

$$(U_m^* * U_n)(y) + (U_m^* * U_n)(-y) = 2\,\Re\left(\frac{\exp(2\pi i m y/L) - \exp(2\pi i n y/L)}{2\pi i(n-m)}\right)$$
$$= \frac{\sin(2\pi m y/L) - \sin(2\pi n y/L)}{\pi(n-m)}. \tag{8}$$

For $m = n$ the same computation gives

$$(U_n^* * U_n)(y) = \frac{1}{-}L\int_y^L \exp(2\pi i n(y-x)/L + 2\pi i n x/L)dx = (1-y/L)\exp(2\pi i n y/L)$$

and

$$(U_n^* * U_n)(y) + (U_n^* * U_n)(-y) = 2\,\Re\left((1-y/L)\exp(2\pi i n y/L)\right)$$
$$= 2(1-y/L)\cos(2\pi n y/L). \tag{9}$$

We can summarize the above computation as follows

Proposition 4.1. *Let $\mathcal{D}$ be as above and Q be the quadratic form Q of (7). Let $\psi(x) := \frac{1}{\pi}\int_0^L \sin(2\pi x(1-y/L))\mathcal{D}(y)dy$. The matrix elements $q_{m,n}$ of Q are given as follows*

$$q_{m,n} = \begin{cases} \frac{\psi(m)-\psi(n)}{m-n} & \text{if } n \neq m, \\ \psi'(n), & \text{if } n = m. \end{cases} \tag{10}$$

Proof. One has, for $n \neq m$, using (7), (8) and the equality $\sin(2\pi x(1-y/L) = -\sin(2\pi x y/L)$ for $x \in \mathbb{Z}$,

$$\langle U_m | U_n \rangle_Q = \int_0^L \left((U_m^* * U_n)(y) + (U_m^* * U_n)(-y)\right)\mathcal{D}(y)dy =$$
$$\int_0^L \frac{\sin(2\pi m y/L) - \sin(2\pi n y/L)}{\pi(n-m)}\mathcal{D}(y)dy = \frac{\psi(n) - \psi(m)}{n-m}.$$

For $n = m$, one has using (7), (9)

$$\langle U_n | U_n \rangle_Q = 2 \int_0^L (1 - y/L) \cos(2\pi n y/L) \mathcal{D}(y) dy = \partial_x \psi(x)|_{x=n}$$

which gives the required equality. □

4.2. The diagonal values. Proposition 4.1 gives the diagonal values of the matrix of the quadratic form Q, but unlike the off-diagonal values which only depend upon the first Fourier components of the distribution $\mathcal{D}$, the diagonal values involve all the Fourier components of $\mathcal{D}$. This follows from the equalities

$$2 \int_0^1 (1 - x) \exp(2\pi i\, kx) \cos(2\pi n x)\, dx = \frac{ik}{\pi k^2 - \pi n^2}, \qquad \forall k \neq \pm n;$$
$$2 \int_0^1 (1 - x) \exp(2\pi i n x) \cos(2\pi n x)\, dx = \frac{1}{2} + \frac{i}{4\pi n}, \qquad \forall n \neq 0.$$

Indeed these equalities show that the Fourier coefficient a_k of $\mathcal{D}(x)$ appears in the diagonal term $q_{n,n}$ as, for $n \neq 0$,

$$q_{n,n} = \sum_{k \neq \pm n} a_k \frac{ik}{\pi k^2 - \pi n^2} + \frac{1}{2}(a_n + a_{-n}) + \frac{1}{4\pi}(\frac{i}{n} a_n - \frac{i}{n} a_{-n}).$$

Since the distribution $\mathcal{D}(x)$ is real valued, one has $a_{-k} = \overline{a}_k$ so that the terms ika_k and $i(-k)a_{-k} = \overline{ika_k}$ add up to a real contribution for $k \neq \pm n$. With $a_k = x_k + i y_k$ for $k > 0$, one gets for $k \neq n$,

$$ika_k + i(-k)a_{-k} = -2k y_k$$

which gives the first contribution to $q_{n,n}$ as

$$\sum_{k>0, k \neq n} y_k \frac{2k}{\pi n^2 - \pi k^2}.$$

For $k = n$ one gets the two terms

$$\frac{1}{2}(a_n + a_{-n}) + \frac{1}{4\pi}(\frac{i}{n} a_n - \frac{i}{n} a_{-n}) = x_n - y_n \frac{1}{2\pi n}.$$

For $n = 0$

$$q_{0,0} = \sum_{k \neq 0} a_k \frac{ik}{\pi k^2} + a_0 = x_0 - \sum_{k>0} y_k \frac{2}{\pi k}.$$

Proposition 4.2. *Let $N \in \mathbb{N}$. The matrices $(q_{i,j})$, $i, j \in \{-N, \ldots, N\}$ obtained from distributions $\mathcal{D}$ by Proposition 4.1, are all matrices of the following form, where a_i and b_j are real numbers with $a_{-i} = a_i$ and $b_{-i} = -b_i$, $\forall i \in \{-N, \ldots, N\}$*

$$q_{i,i} = a_i, \quad \forall i, \qquad q_{i,j} = \frac{b_i - b_j}{i - j}, \quad \forall j \neq i; \qquad i, j \in \{-N, \ldots, N\}. \tag{11}$$

Moreover given a matrix $Q = (q_{i,j})$, $i, j \in \{-N, \ldots, N\}$ of the above form there exists a unique real distribution $\mathcal{D}(x)$ all of whose Fourier components $a_n = 0$ for $n \notin \{-N, \ldots, N\}$ and whose associated matrix is Q.

Proof. Proposition 4.1 shows that the matrices $(q_{i,j})$, $i, j \in \{-N, \ldots, N\}$ obtained from distributions $\mathcal{D}$ by Proposition 4.1 are of the form given by (11). Let us show that conversely any matrix of the form (11) appears.

The equality $\psi(x) = \frac{1}{\pi}\int_0^L \sin(2\pi x(1-y/L))\mathcal{D}(y)dy$ of Proposition 4.1, gives, in terms of the Fourier coefficients $a_k = x_k + iy_k$ of $\mathcal{D}(x)$, for $n \in \mathbb{Z}$, taking $L = 1$ for simplicity,

$$\psi(n) = -\frac{1}{\pi}\int_0^1 \sin(2\pi ny)\mathcal{D}(y)dy = \frac{1}{2\pi i}a_n - \frac{1}{2\pi i}a_{-n} = \frac{1}{\pi}y_n.$$

Thus the matrix entries $q_{i,j}$ for $i \neq j$ determine the real numbers y_j which are the imaginary parts of the Fourier coefficients a_k of $\mathcal{D}$. These imaginary parts then give the following contributions to the diagonal value, for $n \neq 0$,

$$q_{n,n} = x_n - y_n\frac{1}{2\pi n} + \sum_{k>0,k\neq n} y_k\frac{2k}{\pi n^2 - \pi k^2} \tag{12}$$

and we can choose the real part x_n of the Fourier coefficients a_n of $\mathcal{D}$ to obtain arbitrary diagonal values a_i as required. Moreover the off-diagonal values determine uniquely the imaginary parts y_n of the Fourier coefficients of $\mathcal{D}(x)$ for $n \in \{-N, \ldots, N\}$ and the diagonal values then determine uniquely the real parts x_n of the Fourier coefficients of $\mathcal{D}(x)$ for $n \in \{-N, \ldots, N\}$. □

Remark 4.3. It is important to take as a starting point a distribution $\mathcal{D}$ on $[0, L]$ and then define the associated quadratic form using (6), rather than starting from an even distribution on $[-L, L]$. For instance the derivative δ_0' of the Dirac distribution at $0 \in [0, L]$ gives rise to a non-zero quadratic form while the associated even distribution obtained by symmetrisation is equal to 0.

4.3. Relation to the spectral action. There is a close analogy between the quadratic form in (11) and the second derivative with respect to perturbations in the spectral action [6], as we will now explain. Suppose we are given a linear self-adjoint operator D in a finite-dimensional Hilbert space, which is assumed to have simple spectrum labeled by $\{\lambda_j\}_{j=-N}^N$, with corresponding eigenvectors $\{e_j\}$. Suppose that we have a real symmetric positive matrix $Q = (q_{ij})$, $i, j \in \{-N, \ldots, N\}$ defined as

$$q_{ij} = \begin{cases} \frac{b_i - b_j}{\lambda_i - \lambda_j} & i \neq j \\ a_i & i = j \end{cases} \tag{13}$$

for some $a_i, b_i \in \mathbb{R}$. The quadratic form Q is given in terms of the Hilbert space inner product as

$$Q(f, g) = \langle Qf \mid g\rangle = \langle f \mid Qg\rangle. \tag{14}$$

Consider now an even smooth function $f : \mathbb{R} \to \mathbb{R}$. Under perturbations $D \mapsto D + A$ it can be computed that the quadratic form given by the second Gâteaux derivative of the spectral action is of the form [13,15,16,19,20]

$$\frac{1}{2}\frac{d^2}{dt^2}(\mathrm{tr}\, f(D + tA))\,|_{t=0} = \sum_{i,j} A_{ij}A_{ji}q_{ij}$$

where q_{ij} is exactly as in (11) for $b_i = f'(\lambda_i)$, $a_i = f''(\lambda_i)$ and $\lambda_i = i$, the spectrum of the Dirac operator $D = D_{S^1}$ on the circle.

5. Finite Dimensional Even Case

In this section we deal with the general finite dimensional even case. We are given a real symmetric positive matrix $Q = (q_{i,j})$, $i, j \in \{-N, \dots, N\}$ of the form (11), *i.e.*

$$q_{i,i} = a_i, \quad \forall i, \qquad q_{i,j} = \frac{b_i - b_j}{i - j}, \quad \forall j \neq i; \qquad i, j \in \{-N, \dots, N\}$$

where the scalars a_i fulfill $a_{-j} = a_j$ and $b_{-j} = -b_j$ for all $j \in \{-N, \dots, N\}$.

We let e_j, $j \in \{-N, \dots, N\}$, be the canonical orthonormal basis given by the vectors $(\delta(k, j))$ whose all components are 0 except one. Using the canonical inner product $\langle \alpha \mid \beta \rangle$, the quadratic form Q is given by

$$Q(f, g) = \langle Qf \mid g \rangle = \langle f \mid Qg \rangle. \tag{15}$$

Lemma 5.1. *(i) Let γ be such that $\gamma(e_j) := e_{-j}$ for all $j \in \{-N, \dots, N\}$. One has $\gamma^2 = Id$ and $Q\gamma = \gamma Q$.*
(ii) Let D be defined by $D(e_j) := j\, e_j$ for all $j \in \{-N, \dots, N\}$. One has $D\gamma = -\gamma D$ and

$$D\,Q - Q\,D = |\beta\rangle\langle\eta| - |\eta\rangle\langle\beta|, \quad \beta = \sum b_j\, e_j, \quad \eta = \sum e_j. \tag{16}$$

Proof. (i) One has $q_{-i,-j} = q_{i,j}$ for all $i, j \in \{-N, \dots, N\}$.
(ii) The diagonal elements of the diagonal matrix D are antisymmetric which gives $D\gamma = -\gamma D$. Let us prove (16). One has $(DQ)_{i,j} = i q_{i,j}$, $(QD)_{i,j} = j q_{i,j}$ so that $(D\,Q - Q\,D)_{i,j} = b_i - b_j$ for all $i, j \in \{-N, \dots, N\}$. Similarly one has

$$(|\beta\rangle\langle\eta|)_{i,j} = |\beta\rangle_i \langle\eta|_j = b_i, \quad (|\eta\rangle\langle\beta|)_{i,j} = |\eta\rangle_i \langle\beta|_j = b_j$$

which gives the required equality. □

Lemma 5.2. *Assume $Q \geq 0$ and $Q\xi = 0$ where $\gamma\xi = \xi$ and $\langle \xi \mid \eta \rangle = 1$.*

(i) One has $Q\,D\,\xi = -\beta$.
(ii) The operator $D' := D - |D\,\xi\rangle\langle\eta|$ is selfadjoint with respect to the inner product defined by Q.

Proof. (i) We apply (16) and get, using $Q\xi = 0$ and $\langle\beta|\xi\rangle = 0$ since the two eigenspaces of γ are orthogonal,

$$-Q\,D\,\xi = (D\,Q - Q\,D)\xi = |\beta\rangle\langle\eta|\xi\rangle - |\eta\rangle\langle\beta|\xi\rangle = \beta.$$

(ii) The inner product defined by Q is given by (15), *i.e.*

$$\langle f \mid g \rangle_Q = \langle Qf \mid g \rangle.$$

Thus we want to show that $\langle D'f \mid g \rangle_Q = \langle f \mid D'g \rangle_Q$ for all f, g. One has, with $R = -|D\,\xi\rangle\langle\eta|$

$$\langle D'f \mid g \rangle_Q = \langle QD'f \mid g \rangle = \langle QDf \mid g \rangle + \langle QRf \mid g \rangle.$$

By (i), one has $QR = -|QD\xi\rangle\langle\eta| = |\beta\rangle\langle\eta|$. Moreover by (16), one has $QD - DQ = -|\beta\rangle\langle\eta| + |\eta\rangle\langle\beta|$. Thus

$$\langle D'f \mid g\rangle_Q = \langle DQf \mid g\rangle + \langle R'f \mid g\rangle, \quad R' = |\eta\rangle\langle\beta|.$$

Moreover, using that both Q and D are selfadjoint,

$$\langle f \mid D'g\rangle_Q = \langle Qf \mid Dg\rangle + \langle Qf \mid Rg\rangle = \langle DQf \mid g\rangle + \langle f \mid QRg\rangle$$

and the required equality follows from

$$\langle f \mid QRg\rangle = \langle f \mid \beta\rangle\langle\eta|g\rangle = \langle R'f \mid g\rangle.$$

□

Lemma 5.3. *Let Q, D, ξ, η and D' be as in Lemma 5.2. Then*
(i) *Let $s \notin \{-N, \dots, N\}$. Then*

$$\mathrm{Det}(D' - s) = 0 \iff \sum_{j=-N}^{N} \frac{\xi_j}{s - j} = 0. \tag{17}$$

(ii) *One has* $\mathrm{Det}(D') = 0$*, and for* $j \in \{-N, \dots, N\}$*,* $j \neq 0$*,* $\mathrm{Det}(D' - j) = 0 \iff \xi_j = 0$.

Proof. *(i)* We start by writing, in terms of $R = -\langle D\xi\rangle\langle\eta|$:

$$D' - s = D + R - s = (D - s)\left(\mathrm{Id} + (D - s)^{-1}R\right)$$

Consequently

$$\mathrm{Det}(D' - s) = \mathrm{Det}(D - s)\mathrm{Det}(\mathrm{Id} + (D - s)^{-1}R).$$

To compute the second determinant we use the identity

$$\mathrm{Det}(\mathrm{Id} + A) = \sum_{k=0}^{\infty} \mathrm{tr}\left(\wedge^k A\right)$$

applied to the rank one operator $A = (D - s)^{-1}R$. The higher exterior powers $\wedge^k A$ vanish for $k > 1$ thus

$$\mathrm{Det}(\mathrm{Id} + (D - s)^{-1}R) = 1 - \mathrm{tr}\left(|(D - s)^{-1}D\xi\rangle\langle\eta|\right) = -s\langle\eta|(D - s)^{-1}\xi\rangle,$$

using $(D - s)^{-1}D\xi = \xi + s(D - s)^{-1}\xi$ and $\langle\eta|\xi\rangle = 1$. Hence,

$$\mathrm{Det}(D' - s) = -s\mathrm{Det}(D - s)\langle\eta|(D - s)^{-1}\xi\rangle = -s \prod_{i=-N}^{N} (i - s) \sum_{j=-N}^{N} (j - s)^{-1}\xi_j. \tag{18}$$

We conclude that if $s \neq j$ for $j = -N, \dots, N$ then $\mathrm{Det}(D' - s) = 0$ iff $\sum_{j=-N}^{N}(j - s)^{-1}\xi_j = 0$.

For *(ii)* we have that $D'\xi = 0$ so $\mathrm{Det}(D') = 0$. For $s = j \neq 0$ we find that the only non-vanishing term on the right-hand side of (18) is $j\prod_{i\neq j}(i - j)\xi_j$ which is zero iff $\xi_j = 0$. □

Remark 5.4. The expression (18) for $\mathrm{Det}(D'-s)$ is related to the ordinary Lagrange interpolation polynomial for the function $f(\lambda)$ at the points $\lambda_0, \lambda_1, \dots, \lambda_n$:

$$P(x) = \sum_{k=0}^{n} f(\lambda_k) \cdot \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x-\lambda_j}{\lambda_k - \lambda_j}.$$

We now compute the Fourier transform of functions on $[0, L]$ translated to $[-\frac{L}{2}, \frac{L}{2}]$ and extended by 0 to the full line $\mathbb{R}$. The Fourier transform is defined by

$$\mathbb{F}(f)(s) := \int_{\mathbb{R}} f(x)\exp(-isx)dx.$$

The next Proposition is a reformulation of the Shannon sampling theorem ([18]).

Proposition 5.5. *Let $f \in L^2([0, L])$ and $f^\sigma(x) := f(x + \frac{L}{2})$ for $|x| \le \frac{L}{2}$ be extended by 0 on $\mathbb{R}$.*
(i) The restriction of the Fourier transform of f^σ to $\frac{2\pi}{L}\mathbb{Z}$ is given by the Fourier transform $\widehat{f}$ of $f \in L^2(\mathbb{R}/L\mathbb{Z})$ as follows

$$\mathbb{F}(f^\sigma)(\frac{2\pi}{L}n) = (-1)^n\, \widehat{f}(n), \qquad \forall n \in \mathbb{Z}. \tag{19}$$

(ii) The Fourier transform of f^σ is given by

$$\mathbb{F}(f^\sigma)(s) = \sin(Ls/2)\sum_{\mathbb{Z}} \widehat{f}(n)\frac{1}{Ls/2 - n\pi}. \tag{20}$$

Proof. (i) Let $n \in \mathbb{Z}$. One has by definition $\widehat{f}(n) = \int_0^L f(x)\exp(-2\pi inx/L)dx$ and

$$\mathbb{F}(f^\sigma)(\frac{2\pi}{L}n) = \int_{\mathbb{R}} f^\sigma(x)\exp(-2\pi inx/L)dx = \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x+\frac{L}{2})\exp(-2\pi inx/L)dx =$$

$$(-1)^n \int_0^L f(x)\exp(-2\pi inx/L)dx = (-1)^n\, \widehat{f}(n).$$

(ii) One has $f(x) = {}^1L \sum_{\mathbb{Z}} \widehat{f}(n)\exp(2\pi inx/L)$, thus it is enough to treat the case $f(x) = \exp(2\pi inx/L)$. Then

$$\mathbb{F}(f^\sigma)(s) = \int_{\mathbb{R}} f^\sigma(x)\exp(-isx)dx = \int_{-\frac{L}{2}}^{\frac{L}{2}} \exp(2\pi in(x+\frac{L}{2})/L)\exp(-isx)dx =$$

$$= (-1)^n \int_{-\frac{L}{2}}^{\frac{L}{2}} \exp(i(\frac{2\pi}{L}n - s)x)dx = (-1)^n \frac{1}{i(\frac{2\pi}{L}n - s)}(\exp(i(\frac{2\pi}{L}n - s)x))_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= 2L\frac{\sin(Ls/2)}{Ls - 2\pi n}$$

which gives (20) using $f(x) = {}^1L \sum_{\mathbb{Z}} \widehat{f}(n)\exp(2\pi inx/L)$. □

After these preliminaries we obtain

Theorem 5.6. *Let Q be a real symmetric positive matrix of the form (11) with one dimensional kernel which is even with respect to γ. Let $\xi \in \ker\ Q$.*
(i) *All the roots of the following polynomial are real:*

$$P(s) = \sum_{k\in\{-N,\dots,N\}} \xi_k \times \left(\prod_{j\in\{-N,\dots,N\}, j\neq k} (j - s) \right). \tag{21}$$

(ii) *The Fourier transform $\widehat{\xi}(z)$ of the function*

$$\xi(x) := \sum \xi_k \exp(2\pi i k x),\ \forall x \in [0, 1],\ \xi(x) = 0,\ \forall x \notin [0, 1]$$

is entire and has all its zeros on the real line.

Proof. *(i)* Recall D, γ from Lemma 5.1 and note that D has one-dimensional kernel. If $e_0 \in \ker\ Q$ one checks (i) and (ii) directly. Thus we assume that $e_0 \notin \ker\ Q$. Let $\xi \in \ker\ Q$, $\xi \neq 0$. The real symmetric positive matrix Q defines an inner product on $\mathbb{R}^{2N+1}$ and its radical consists of the one dimensional subspace generated by ξ. One has $D\xi \neq 0$ since otherwise $e_0 \in \ker\ Q$. One has $QD\xi \neq 0$ since $D\xi$ is odd and therefore linearly independent of ξ while $\ker\ Q$ is one-dimensional. By (16) one has

$$0 \neq (D\,Q - Q\,D)(\xi) = |\beta\rangle\langle\eta|\xi\rangle - |\eta\rangle\langle\beta|\xi\rangle = |\beta\rangle\langle\eta|\xi\rangle.$$

Thus one can normalize ξ so that $\langle\eta|\xi\rangle = 1$. Let then $D' := D - |D\,\xi\rangle\langle\eta|$ as in Lemma 5.2. One has $D'(\xi) = 0$ and thus D' induces an operator $D"$ on the Euclidean space E obtained as the separated quotient of $(\mathbb{R}^{2N+1}, Q)$. By Lemma 5.2, (ii), the operator $D"$ is selfadjoint in E. Thus the real spectral theorem (see [3], Theorem 7.29) applies and shows that the characteristic polynomial of $D"$ has all its roots in $\mathbb{R}$. Let v_j be an orthonormal basis of E of eigenvectors for $D"$ with eigenvalues λ_j. Let $w_j \in \mathbb{R}^{2N+1}$ be lifts of the v_j. One has $D"(v_j) = \lambda_j v_j$ and hence $D'(w_j) = \lambda_j w_j + s_j\xi$ for some real scalars s_j. Thus in the basis of $\mathbb{R}^{2N+1}$ formed by ξ and the w_j, the matrix of D' is triangular, with 0 and the λ_j on the diagonal. Thus one gets

$$\mathrm{Det}(D' - s) = -s \prod (\lambda_j - s).$$

Comparing this formula with (18) one obtains that the polynomial $P(s)$ of (21) has all its zeros in $\mathbb{R}$.
(ii) The Fourier transform of the function with support in $[0, 1]$ given there by $\exp(2\pi i k x)$ is

$$\int_0^1 \exp(2\pi i k x) \exp(-isx)\, dx = \frac{2e^{-\frac{is}{2}} \sin\left(\frac{s}{2}\right)}{s - 2\pi k}.$$

Thus the Fourier transform of $\xi(x)$ is

$$\widehat{\xi}(z) = 2e^{-\frac{iz}{2}} \sin\left(\frac{z}{2}\right) \left(\sum_{\{-N,\dots,N\}} \frac{\xi_j}{z - 2\pi j} \right).$$

The zeros $z \in 2\pi\mathbb{Z}$ of $\sin\left(\frac{z}{2}\right)$ cancel the pole at $2\pi j$ which occurs when $\xi_j \neq 0$ and remain as zeros of $\widehat{\xi}(z)$ otherwise. The other zeros are given by the roots of $P(z/2\pi) = 0$ where $P(z)$ is defined in (21). Thus by (i), all these zeros are real. □

5.1. General finite dimensional operator D. Consider as before the more general situation of a linear self-adjoint operator D in a finite-dimensional Hilbert space, which is assumed to have simple spectrum labeled by $\{\lambda_j\}_{j=-N}^{N}$, with corresponding eigenvectors $\{e_j\}$. We then consider the real symmetric positive matrix $Q = (q_{ij})$, $i, j \in \{-N, \dots, N\}$ defined in (13).

The following Lemma's are the analogues of Lemmas 5.1, 5.2 and 5.3, whose proofs follow *mutatis mutandis*.

Lemma 5.7. *Suppose that* $\lambda_{-i} = -\lambda_i$ *and* $b_{-i} = -b_i$ *for all* $i \in \{-N, \dots, N\}$.

(i) Let γ *be such that* $\gamma(e_i) := e_{-i}$ *for all* $i \in \{-N, \dots, N\}$. *One has* $\gamma^2 = Id$ *and* $Q\gamma = \gamma Q$.
(ii) One has $D\gamma = -\gamma D$ *and*

$$D\,Q - Q\,D = |\beta\rangle\langle\eta| - |\eta\rangle\langle\beta|, \quad \beta = \sum b_i\, e_i, \quad \eta = \sum e_i, \tag{22}$$

so that β *is odd and and* η *is even with respect to the* $\mathbb{Z}_2$*-grading given by* γ.

Lemma 5.8. *Let* D, Q, γ *be as in Lemma 5.7, assume* $Q \geq 0$ *and* $Q\xi = 0$ *where* $\gamma\xi = \xi$ *and* $\langle\xi \mid \eta\rangle = 1$.
(i) One has $Q\,D\,\xi = -\beta$.
(ii) The operator $D' := D - |D\,\xi\rangle\langle\eta|$ *is self-adjoint with respect to the inner product defined by* Q.

Lemma 5.9. *Let* Q, D, ξ, η *and* D' *be as in Lemmas 5.7 and 5.8, and assume that* D *has simple spectrum. Then*

(i) Let $s \in \mathbb{C}\backslash\{-\lambda_N, \dots, \lambda_N\}$ *and* $s \neq 0$. *Then*

$$\operatorname{Det}(D' - s) = 0 \iff \sum_{j=-N}^{N} \frac{\xi_j}{s - \lambda_j} = 0. \tag{23}$$

(ii) One has $\operatorname{Det}(D') = 0$, *and if* $\lambda_j \neq 0$ *we have* $\operatorname{Det}(D' - \lambda_j) = 0 \iff \xi_j = 0$.

As a result, we also have the following analogue of Theorem 5.6(i):

Proposition 5.10. *Let* D *have simple spectrum and let* Q *be a real symmetric positive matrix of the form (13) with one dimensional even kernel. Let* $\xi \in \ker\, Q$. *Then all the roots of the following polynomial are real:*

$$P(s) = \sum_{k \in \{-N, \dots, N\}} \xi_k \times \left(\prod_{j \in \{-N, \dots, N\}, j \neq k} (\lambda_j - s) \right). \tag{24}$$

We then obtain Theorem 5.6(ii) as a corollary to this result, when it is applied to the case $D = D_{S^1}$ so that $\lambda_j = j$.

6. Infinite Dimensional Case

In this section we prove the result announced in the Introduction,

Theorem 6.1. *Let $L > 0$, and let $\mathcal{D}$ be a real distribution on the interval $[0, L]$. Let Q be the quadratic form defined on trigonometric polynomials by (6). Assume that Q defines a lower-bounded essentially selfadjoint operator and that the minimum of its spectrum is a simple, isolated eigenvalue λ, with even eigenfunction[1] ξ. Then all the zeros of the entire function $\widehat{\xi}(z)$, $z \in \mathbb{C}$, the Fourier transform of ξ lie on the real line.*

Proof. By hypothesis the trigonometric polynomials form a core for the operator A which is defined by

$$\langle \alpha \mid A\beta \rangle = \langle \alpha \mid \beta \rangle_Q.$$

We normalize the eigenvector ξ by $\|\xi\| = 1$. Given $\epsilon > 0$ there exists an even trigonometric polynomial η_ϵ with

$$\|\eta_\epsilon\| = 1, \quad \|\xi - \eta_\epsilon\| < \epsilon, \quad \langle \eta_\epsilon \mid \eta_\epsilon \rangle_Q < \lambda + \epsilon. \tag{25}$$

Let N be a finite integer such that the support of η_ϵ is contained in $\{-N, \dots, N\}$. By Proposition 4.1 the matrix of the restriction Q_N of the quadratic form Q to the space E_N of trigonometric polynomials with support in $\{-N, \dots, N\}$ is of the form (11). Since Q_N is a restriction of Q to a subspace, its minimum is $\geq \lambda$ and the above inequalities show that it is between λ and $\lambda + \epsilon$. By hypothesis the spectrum of the operator A is contained, except for the simple eigenvalue λ in the interval $[\lambda + \delta, \infty)$ for some $\delta > 0$. Thus the restriction of Q to the orthogonal complement of ξ fulfills

$$\langle \alpha \mid \alpha \rangle_Q \geq (\lambda + \delta)\|\alpha\|^2, \quad \forall \alpha \mid \langle \alpha \mid \xi \rangle = 0.$$

This holds in particular in the codimension one subspace F_N of E_N defined by $\langle \alpha \mid \xi \rangle = 0$ (note that F_N cannot be all of E_N since then we also would have had $\xi \perp \eta_\epsilon$ which contradicts (25)). The smallest eigenvalue of Q_N fulfills $\lambda \leq \lambda_N \leq \lambda + \epsilon$. By the min-max theorem, the next eigenvalue $\mu_N \geq \lambda_N$ of Q_N is given by

$$\mu_N = \max_{\substack{\mathcal{M} \subset E_N \\ \operatorname{codim}(\mathcal{M}) = 1}} \; \min_{x \in \mathcal{M}, \|x\| = 1} Q_N(x)$$

so that, using $\mathcal{M} = F_N$ one gets $\mu_N \geq \lambda + \delta$. This shows that for $\epsilon < \delta/2$ the eigenvalue λ_N of Q_N is simple and the only one in the interval $[\lambda, \lambda + \delta]$. Let then P be the spectral projection of Q_N for the eigenspace associated to the minimal eigenvalue λ_N. Decomposing

$$\eta_\epsilon = P\eta_\epsilon + (1 - P)\eta_\epsilon = \alpha + \beta \;\Rightarrow\; Q(\eta_\epsilon) = \lambda_N \|\alpha\|^2 + Q(\beta)$$

where $Q(\beta) \geq (\lambda + \delta)\|\beta\|^2$. Thus by (25), one gets that the convex combination with weights $\|\alpha\|^2$ and $\|\beta\|^2$ of λ and $Q(\beta)/\|\beta\|^2 \geq (\lambda + \delta)$ is less than $\lambda + \epsilon$. It follows that $\|\beta\|^2 = 1 - \|\alpha\|^2 \leq \epsilon/\delta$. Let then ξ_N be the eigenvector of Q_N for the eigenvalue λ_N given by $P\eta_\epsilon$. One controls

$$\|\eta_\epsilon - P\eta_\epsilon\| \leq \sqrt{\epsilon/\delta}, \quad \|\xi - \eta_\epsilon\| < \epsilon \Rightarrow \|\xi - \xi_N\| \leq \epsilon + \sqrt{\epsilon/\delta}.$$

[1] *i.e.* invariant under the symmetry $x \mapsto L - x$ of $[0, L]$

Since the even functions form a closed subspace in $L^2[0, L]$, and ξ is even, for ϵ small enough the vector ξ_N is even as well. Thus, it follows from Theorem 5.6 that all the zeros of the Fourier transform $\widehat{\xi}_N$ are real. Moreover when $\epsilon \to 0$ the vectors ξ_N converge in norm to ξ so that the sequence $\widehat{\xi}_N(z)$ converges to $\widehat{\xi}(z)$ uniformly on compact subsets in $\mathbb{C}$ (as in the proof of Theorem 3.1). But then the Hurwitz theorem applies, allowing us to conclude that all zeros of $\widehat{\xi}(z)$ are real. □

7. Spectral Action and Divided Differences

As already observed there is a close relation of the quadratic form in (15) and the spectral action $\operatorname{tr} f(D)$ introduced in [6]. We will now analyze this in more detail for perturbations of the type $D \mapsto D + R$ with $R = -|D\xi\rangle\langle\eta|$ as in Lemma 5.2, extending the Taylor expansions derived in [13,15,16,19,20] to this case.

First, in order to make sense of the spectral action for (not necessarily self-adjoint or normal) bounded perturbations of a self-adjoint operator we write $f(x)$ as a Fourier transform, and then invoke Araki's expansionals [1]—or Dyson series—to give meaning to $e^{i(H_0+V)}$ for bounded perturbation V of a self-adjoint operator H_0 (to be precise, this is [1, Eq. 5.16]):

$$\begin{aligned} e^{i(H_0+V)} &:= \operatorname{Exp}_r\left(\int_0^1 ; ie^{isH_0} V e^{-isH_0} ds\right) e^{iH_0} \\ &:= \sum_{n\geq 0} i^n \int_{\Delta_n} e^{is_0H_0} V e^{is_1H_0} \cdots V e^{is_nH_0} d^n s. \end{aligned} \tag{26}$$

where the n-simplex Δ_n is parametrized by tuples $(s_0, \ldots, s_n) \in \mathbb{R}^{n+1}_{\geq 0}$ satisfying $\sum_k s_k = 1$. In our case of interest, these expansionals are given by series of the following form:

$$e^{i\xi(D+tR)} := \sum_{n\geq 0} (it\xi)^n \int_{\Delta_n} e^{is_0\xi D} R e^{is_1\xi D} \cdots R e^{is_n\xi D} d^n s. \tag{27}$$

Note that since $|\Delta_n| = 1/n!$ the n'th summand in the expansional is norm bounded by $t^n|\xi|^n\|R\|^n/n!$ so that the series expansion is norm convergent. This suggest to define for suitable functions f:

$$f(D + tR) := \int_{\mathbb{R}} \widehat{f}(\xi) e^{i\xi(D+tR)} d\xi. \tag{28}$$

More precisely, we have

Lemma 7.1. *Let D be a self-adjoint operator on $\mathcal{H}$, R bounded operator on $\mathcal{H}$ and let f be such that $\|\widehat{f^{(n)}}\|_1 \leq C^{n+1} n!$ for all $n \geq 0$ and some $C \geq 1$. Then for sufficiently small t the expression in (28) is a bounded operator on $\mathcal{H}$.*

Proof. We estimate:

$$\|f(D + tR)\| \leq \sum_{n\geq 0} \frac{t^n \|R\|^n}{n!} \int_{\mathbb{R}} |\widehat{f}(\xi)\xi^n| d\xi \leq \frac{C}{1 - tC\|R\|}$$

which is bounded for $t < 1/(C\|R\|)$. □

In order to define the spectral action as the trace of this operator, we need a more restrictive class of functions. Namely, as in [16] we define

$$\mathcal{E}^s := \left\{ f \in C^\infty : \text{ there exists } C \geq 1 \text{ s.t. } \|(\widehat{fu^m})^{(n)}\|_1 \leq C^{n+1} n! \text{ for all } m \leq s \text{ and } n \geq 0 \right\},$$

where $u(x) = x - i$.

Lemma 7.2. *If D is s-summable,* i.e. $(D-i)^{-s}$ *is trace-class for some* $s \geq 0$, *and* $f \in \mathcal{E}^s$ *then* $f(D+tR)$ *is trace-class for sufficiently small* t.

Proof. The proof of the required estimates follows line-by-line the proof of [16, Theorem 6] after having given the meaning (27) to the exponentials appearing in the multiple operator integrals (*i.e.* Definition 2 in *loc. cit.*). □

We recall the definition of divided differences. Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0, x_1, \ldots x_n$ be distinct points in $\mathbb{R}$. The divided difference of order n is defined by the recursive relations

$$f[x_0] = f(x_0),$$
$$f[x_0, x_1, \ldots x_n] = \frac{f[x_1, \ldots x_n] - f[x_0, x_1, \ldots x_{n-1}]}{x_n - x_0}.$$

Also note the following useful representation, due to Hermite [14]: for any $x_0, \ldots, x_n \in \mathbb{R}$,

$$f[x_0, x_1, \ldots, x_n] = \int_{\Delta_n} f^{(n)}(s_0 x_0 + s_1 x_1 + \cdots + s_n x_n)\, d^n s.$$

This also allows to extend the definition of divided difference to coinciding points.

Lemma 7.3. *Let D be a self-adjoint operator in $\mathcal{H}$ such that $(D-i)^{-s}$ is trace class for some $s \geq 0$, R is a bounded operator in $\mathcal{H}$ and $f \in \mathcal{E}^s$. Then $t \mapsto \operatorname{tr} f(D+tR)$ is smooth in a neighborhood of* 0 *with n'th derivative*

$$\frac{d^n}{dt^n} \operatorname{tr} f(D+tR)|_{t=0} = n! \sum R_{i_1 i_2} \cdots R_{i_n i_1} f'[\lambda_{i_1}, \ldots, \lambda_{i_n}].$$

in terms of the eigenvalues λ_i of D.

Proof. Since $f(D+tR)$ is defined in terms of Araki's expansional formula, we have

$$\begin{aligned}
\frac{d^n}{dt^n} \operatorname{tr} f(D+tR)|_{t=0} &= \int \frac{d^n}{dt^n} \operatorname{tr}(\widehat{f}(\xi) e^{i\xi(D+tR)})|_{t=0} d\xi \\
&= n! \sum_{i_1, i_2, \ldots, i_n} R_{i_1 i_2} \cdots R_{i_n i_1} \int (i\xi)^n \exp[i\xi\lambda_{i_1}, \ldots, i\xi\lambda_{i_n}, i\xi\lambda_{i_1}] \widehat{f}(\xi) d\xi \\
&= n! \sum_{i_1, i_2, \ldots, i_n} R_{i_1 i_2} \cdots R_{i_n i_1} f[\lambda_{i_1}, \ldots, \lambda_{i_n}, \lambda_{i_1}].
\end{aligned}$$

□

Remark 7.4. It would be interesting to extend the above definition of the spectral action for not necessarily self-adjoint perturbations to the case where also the assumption on self-adjointness on the operator D is relaxed. This has potential applications to Lorentzian spectral triples, Let us now take a function f such that $f''(\lambda_j) = a_j$ and $f'(\lambda_j) = b_j$, where a_j, b_j are the coefficients of the quadratic form as in 13. We then have

$$\frac{d}{dt}\mathrm{tr}\, f(D + tR)|_{t=0} = \sum R_{jj} b_j; \qquad \frac{d^2}{dt^2}\mathrm{tr}\, f(D + tR)|_{t=0} = \sum R_{ij} R_{ji} q_{ij}.$$

Proposition 7.5. *Let $\mathcal{H}$ be finite-dimensional. In the notation of Lemma 5.7, assume $Q = (q_{ij}) \geq 0$ and $Q\xi = 0$ where $\gamma\xi = \xi$ and $\langle \xi | \eta \rangle = 1$. Let $R = -|D\xi\rangle\langle\eta|$ so that $R_{ij} = -(D\xi)_i$. Then we have*

$$\frac{d}{dt}\mathrm{tr}\, f(D + tR)|_{t=0} = \langle D\xi, D\xi \rangle_Q; \qquad \frac{d^2}{dt^2}\mathrm{tr}\, f(D + tR)|_{t=0} = \langle D\xi, D\xi \rangle_Q.$$

Proof. The second derivative is reduced to the first derivative because $QD\xi = -\beta$ (*cf.* Lemma 5.8(i)). Indeed, this yields:

$$\sum_j R_{ji} f'[\lambda_i, \lambda_j] = -\sum_j f'[\lambda_i, \lambda_j](D\xi)_j = -(QD\xi)_i = b_i = f'(\lambda_i).$$

From this it follows that

$$\sum R_{ij} R_{ji} f'[\lambda_i, \lambda_j] = \sum_i R_{ii} f'(\lambda_i) = -\sum_i (D\xi)_i f'(\lambda_i)$$
$$= \sum_i (D\xi)_i \overline{(QD\xi)_i} = \langle D\xi, D\xi \rangle_Q.$$

□

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Declarations

Conflict of interest statement The authors declare that there is no Conflict of interest.

Appendix A. An instance of truncation matrices

In this appendix we describe an example where the truncation matrices admit simple maximal and minimal eigenvalues but this property fails in the limit where the maximal and minimal eigenvalues have multiplicity 2. We let $L = 1$ and take the distribution $\mathcal{D}$ of the form

$$\mathcal{D}(x) = \delta_0(x) + 2\pi\, b \sin(2\pi x). \tag{29}$$

We then compute $\psi(x) := \frac{1}{\pi}\int_0^1 \sin(2\pi x(1-y))\mathcal{D}(y)dy$ as

$$\psi(x) = \frac{1}{\pi}\sin(2\pi x) + 2b\int_0^1 \sin(2\pi y)\sin(2\pi x(1-y))\,dy = \frac{1}{\pi}\sin(2\pi x) + b\frac{\sin(2\pi x)}{\pi - \pi x^2}$$
$$= \frac{\sin(2\pi x)}{\pi}\left(1 + \frac{b}{1-x^2}\right).$$

Thus one has $\psi(n) = 0$ for all $n \in \mathbb{Z}$, except for $n = \pm 1$. Moreover one gets $\psi(-1) = b$ and $\psi(1) = -b$. The derivative is

$$\psi'(x) = \frac{2b\,x\sin(2\pi x)}{\pi\left(1-x^2\right)^2} + \frac{2b\cos(2\pi x)}{1-x^2} + 2\cos(2\pi x).$$

One finds that for $n \in \mathbb{Z}$, not equal to ± 1, one has

$$\psi'(n) = 2 + \frac{2b}{1-n^2}$$

while for $n = \pm 1$ one has $\psi'(n) = \frac{b}{2} + 2$.

Lemma A.1. *Let $L = 1$ and $\mathcal{D}$ be given by (29), Q be the quadratic form Q of (10). The matrix elements $q_{n,m}$ of Q are given by $q_{n,m} = 2\delta_{n,m} + b\,\mu_{n,m}$, where the matrix μ is independent of b and given by $\mu_{n,m} = 0,\ \forall n, m \notin \{-1, 0, 1\},\ n \neq m$ and*

$$(\mu_{n,m})_{n,m\in\{-1,0,1\}} = \begin{pmatrix} \frac{1}{2} & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & \frac{1}{2} \end{pmatrix}, \quad \mu_{n,n} = \frac{2}{1-n^2},\ \forall n \notin \{-1, 0, 1\},$$

$$\mu_{n,-1} = \mu_{-1,n} = \frac{1}{-1-n},\ \forall n \notin \{-1, 0, 1\},\ \mu_{n,1} = \mu_{1,n} = \frac{1}{-1+n},\ \forall n \notin \{-1, 0, 1\}.$$

Proof. This follows from Proposition 4.1 and the above determination of $\psi(n)$ and $\psi'(n)$. □

The matrix elements $\mu_{n,m}$ of μ, for $|n| \leq 4$ and $|m| \leq 4$ are the following

$$\begin{pmatrix}
-\frac{2}{15} & 0 & 0 & \frac{1}{3} & 0 & -\frac{1}{5} & 0 & 0 & 0 \\
0 & -\frac{1}{4} & 0 & \frac{1}{2} & 0 & -\frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & -\frac{2}{3} & 1 & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{2} & 1 & \frac{1}{2} & -1 & -1 & -\frac{1}{3} & -\frac{1}{4} & -\frac{1}{5} \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
-\frac{1}{5} & -\frac{1}{4} & -\frac{1}{3} & -1 & -1 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{3} \\
0 & 0 & 0 & -\frac{1}{3} & 0 & 1 & -\frac{2}{3} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{4} & 0 & \frac{1}{2} & 0 & -\frac{1}{4} & 0 \\
0 & 0 & 0 & -\frac{1}{5} & 0 & \frac{1}{3} & 0 & 0 & -\frac{2}{15}
\end{pmatrix}.$$

Let then ξ be the vector with coordinates $\xi(n) = \frac{1}{1+n}$ for $n \neq -1$ while $\xi(-1) = 0$, and η with $\eta(n) = \frac{1}{-1+n}$ for $n \neq 1$, while $\eta(1) = 0$. Also let the U_n form the canonical orthonormal basis. Thus one has

$$\xi = \sum_{n\neq -1} \frac{1}{1+n} U_n, \quad \eta = \sum_{n\neq 1} \frac{1}{-1+n} U_n.$$

The functions corresponding to these vectors are

$$\xi(x) = -2\pi i \left(x - \frac{1}{2}\right) \exp(-2\pi i x), \quad \eta(x) = -2\pi i \left(x - \frac{1}{2}\right) \exp(2\pi i x).$$

One has $\overline{\xi(x)} = -\eta(x)$

$$\sum_{-k}^{k} \xi(n)\eta(n) = \frac{k^2 - 3k - 2}{2k(k+1)}, \quad \langle \xi \mid \eta \rangle = \frac{1}{2}.$$

We consider the rank 4 matrix given by

$$R := |\eta\rangle\langle U_1| + |U_1\rangle\langle \eta| - |\xi\rangle\langle U_{-1}| - |U_{-1}\rangle\langle \xi|.$$

Its matrix elements $R_{n,m}$ fulfill $R_{n,m} = 0,\ \forall n, m \notin \{-1, 1\}$, while

$$R_{n,1} = R_{1,n} = \eta(n), \quad R_{n,-1} = R_{-1,n} = -\xi(n), \quad \forall n \notin \{-1, 1\}.$$

The corresponding Schwartz kernel $r(x, y)$ is given by

$$r(x, y) = \eta(x)\exp(-2\pi i y) - \exp(2\pi i x)\xi(y) - \xi(x)\exp(2\pi i y) + \exp(-2\pi i x)\eta(y).$$

Lemma A.2. (i) *Let D be the diagonal matrix with diagonal elements $d_n = \frac{2}{1-n^2}$ for $n^2 \neq 1$ and $d_n = \frac{1}{2}$ for $n^2 = 1$. One then has $\mu = D + R$.*
(ii) *The operator D is given by the convolution among periodic functions with period* 1 *by the function*

$$\alpha(x) := -4\pi(x - \frac{1}{2})\sin(2\pi x), \quad \forall x \in [0, 1).$$

Proof. (i) This follows from Lemma A.1.
(ii) One has, for $n \in \mathbb{Z}$, $n \neq \pm 1$,

$$2\pi \int_0^1 \left(x - \frac{1}{2}\right) \sin(2\pi x)\cos(2\pi n x)\, dx = \frac{1}{n^2 - 1}.$$

while the value of this integral is $-\frac{1}{4}$ for $n = \pm 1$. □

Note that the function $\alpha(x)$ needs to be viewed as a periodic function of period 1 and this requires reinterpreting the term $x - \frac{1}{2}$ as $x - E(x) - \frac{1}{2}$ where $E(x)$ is the integer part of $x \in \mathbb{R}$. This plays a role in the formula for the convolution, which is

$$D(f)(x) = \int_0^1 \alpha(x - y) f(y) dy.$$

One has

$$\sin(2\pi(x - y)) = -\cos(2\pi x)\sin(2\pi y) + \sin(2\pi x)\cos(2\pi y)$$

so that

$$\alpha(x-y) = -4\pi(x-y-E(x-y)-\frac{1}{2})\,(-\cos(2\pi x)\sin(2\pi y)+\sin(2\pi x)\cos(2\pi y))$$

and the only term which does not separate as a product of functions of x by a function of y is the term involving $E(x-y)$ which is equal to 0 if $y \le x$ and to -1 if $y > x$. In fact it is more symmetric to add the $\frac{1}{2}$ to $E(x-y)$ which coincides with $\frac{1}{2}\mathrm{Sign}(x-y)$. The other contributions are given by the 4 terms which are, up to the overall factor -2π,

$$(-i)xe^{-2i\pi(y-x)} + ixe^{2i\pi(y-x)} - iye^{2i\pi(y-x)} + iye^{-2i\pi(y-x)}$$

which, taking into account the factor -2π can be rewritten as

$$-\eta(x)\exp(-2i\pi y) + \xi(x)\exp(2i\pi y) - \exp(-2i\pi x)\eta(y) + \exp(2i\pi x)\xi(y)$$

We thus see that these terms cancell the rank 4 additional contribution R and thus we get the following simple formula for the Schwartz kernel $\mu(x, y)$ of the operator μ,

Proposition A.3. (i) *The operator* μ *is given by the formula*

$$\mu(f)(x) = 2\pi \int_0^1 \mathrm{Sign}(x-y)\sin(2\pi(x-y))f(y)dy.$$

(ii) *In general the Schwartz kernel of the operator associated to the distribution* $\mathcal{D}$ *is equal to the restriction to* $x, y \in [0, L]$ *of* $\mathcal{D}(|x-y|)$.

Proof. (i) Follows from the above computation.
(ii) By (7) one has for smooth f, g with support in $[0, L]$,

$$Q(f, g) = \int_0^L \big((g^* * f)(y) + (g^* * f)(-y)\big)\,\mathcal{D}(y)dy = \int_{-L}^L (g^* * f)(y)\mathcal{D}(|y|)dy$$

where one needs to be careful in doubling the coefficient of δ_0 in $\mathcal{D}(|y|)$. This formula does not change if one replaces $f(x)$ by $f(x+\frac{1}{2})$ and $g(x)$ by $g(x+\frac{1}{2})$, shifting their supports to the symmetric interval $[-\frac{L}{2}, \frac{L}{2}]$. Then $Q(f, g) = 0$ when f and g have opposite parity. One then uses the formula for three functions f, g, h, h even

$$\int_{-L}^L (g^* * f)(y)h(-y)dy = \int_{x+y+z=0} g^*(x)f(y)h(z)\omega = \int_{x=y+z} \overline{g(x)}f(y)h(z)dxdy$$

$$= \int \overline{g(x)} \int k(x, y)f(y)dydx, \quad k(x, y) = h(x-y)$$

where ω is the measure on the plane $x+y+z=0$ given by $|dx \wedge dz| = |dx \wedge dy|$. $\square$

One can then investigate numerically the zeros of the polynomials $P_n^{\pm}(s)$ associated by (21) to the eigenvectors for the largest and smallest eigenvalues of the matrix $\mu(n)$ of size $2n+1$, which is the compression of the matrix μ,

$$\mu(n)_{i,j} := \mu_{i,j}, \ \ \forall i, j \in \{-n, \ldots, n\}.$$

For $n = 1$, this matrix is simply

$$\mu(1) = \begin{pmatrix} \frac{1}{2} & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & \frac{1}{2} \end{pmatrix}.$$

Its three eigenvalues are

$$\left\{ \frac{1}{4}\left(\sqrt{57}+3\right), \frac{3}{2}, \frac{1}{4}\left(3-\sqrt{57}\right) \right\}$$

and the corresponding eigenvectors are

$$\begin{pmatrix} 1 & -\frac{3-\sqrt{57}}{\sqrt{57}-9} & 1 \\ -1 & 0 & 1 \\ 1 & -\frac{-\sqrt{57}-3}{\sqrt{57}+9} & 1 \end{pmatrix}.$$

The polynomials $P_1^{\pm}(s)$ are given by

$$P_1^+(s) = 3\left(\sqrt{57}-7\right)s^2 - \sqrt{57}+3, \qquad P_1^-(s) = 3\left(\sqrt{57}+7\right)s^2 - \sqrt{57}-3$$

and their roots are real and are algebraic numbers.
The numerical study of the eigenvalues and eigenvectors of $\mu(n)$ indicates that the largest and smallest eigenvalues are simple for finite n, and that in the limit when $n \to \infty$ the following occurs:

Fact A.4. (i) *The functions realizing the maximum of the matrix* μ *are*

$$f^+(x) = \sin(\pi x), \;\; f^-(x) = \cos(\pi x), \;\; \forall x \in [0, 1].$$

(ii) *The functions realizing the minimum of the matrix* μ *are*

$$g^+(x) = \sin(3\pi x), \;\; g^-(x) = \cos(3\pi x), \;\; \forall x \in [0, 1].$$

One can then deduce the relevant properties of the Fourier transforms as follows.

Proposition A.5. (i) *The Fourier transforms* $h^{\pm}$ *of the functions* $f^{\pm}(x+\frac{1}{2})$ *are given by*

$$h^+(s) = \frac{2\pi \cos\left(\frac{s}{2}\right)}{\pi^2 - s^2}, \;\; h^-(s) = \frac{2\,i\,s\cos\left(\frac{s}{2}\right)}{\pi^2 - s^2}.$$

(ii) *The Fourier transforms* $k^{\pm}$ *of the functions* $g^{\pm}(x+\frac{1}{2})$ *are given by*

$$k^+(y) = \frac{6\pi\cos\left(\frac{y}{2}\right)}{9\pi^2 - y^2}, \;\; k^-(y) = -\frac{2\,i\,y\cos\left(\frac{y}{2}\right)}{y^2 - 9\pi^2}.$$

(iii) *The functions* $h^{\pm}, k^{\pm}$ *are entire functions all of whose zeros are real, and given by all odd multiples of* π *except* $\pm\pi$ *for* h^+, $\pm 3\pi$ *for* k^+ *and including* 0 *for* h^- *and* k^-.
(iv) *The maximal and minimal eigenvalues of the matrix* μ *are* $\frac{8}{3}$ *and* $-\frac{8}{5}$.

Appendix B. Explicit checks for N $\mathcal{D}$ 1, 2

We give concrete proofs of Theorem 5.6 in the simplest cases $N = 1, 2$.

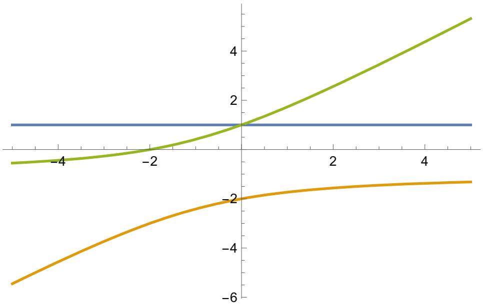
B.1. Case $N = 1$. For $N = 1$ it is enough to treat the following matrix $M(c)$ for $c \in \mathbb{R}$,

$$M(c) = \begin{pmatrix} 0 & -1 & -1 \\ -1 & c & -1 \\ -1 & -1 & 0 \end{pmatrix}. \tag{30}$$

The three eigenvalues are

$$\left\{1, \frac{1}{2}\left(-\sqrt{c^2+2c+9}+c-1\right), \frac{1}{2}\left(\sqrt{c^2+2c+9}+c-1\right)\right\}$$

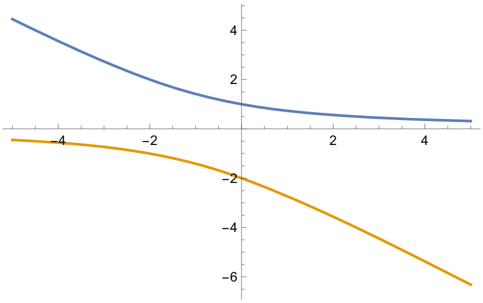
and their dependence on c is as follows:



The corresponding eigenvectors are

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & X(c) & 1 \\ 1 & Y(c) & 1 \end{pmatrix}, \quad X(c) = -\frac{-\sqrt{c^2+2c+9}+c-3}{\sqrt{c^2+2c+9}+c+3}, \quad Y(c) = -\frac{-\sqrt{c^2+2c+9}-c+3}{\sqrt{c^2+2c+9}-c-3}$$

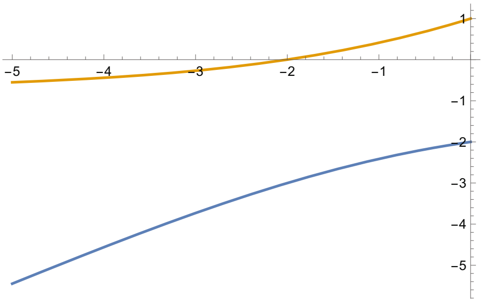
and the dependence of $X(c)$, $Y(c)$ on c is as follows:



The Fourier transform of the vector $(1, x, 1)$ is given by the function of s equal to $s^2(-x) - 2s^2 + 4\pi^2 x$, and thus its roots are real exactly when $x(x+2) \geq 0$. This fails when $-2 < x < 0$, and this is realized by $Y(c)$ for $c < 0$. But the corresponding eigenvalue $y(c)$ fulfills for $c < 0$

$$\frac{1}{2}\left(-\sqrt{c^2+2c+9}+c-1\right) = x(c) < y(c) = \frac{1}{2}\left(\sqrt{c^2+2c+9}+c-1\right) < 1$$

as shown by the graphs for $c < 0$, where $x(c)$ appears in blue, and $y(c)$ remains between $x(c)$ and 1.



We thus get

Proposition B.1. *For any distribution $\mathcal{D}$ the* 3×3 *matrix* $q_{n,m}$ *where* $n, m \in \{-1, 0, 1\}$ *has the property that the zeros of the Fourier transforms of the eigenvectors corresponding to the extremal eigenvalues are real.*

Proof. Note first that it is enough to prove the result for the specific matrix $M(c)$ of (30). Indeed in general the 3×3 matrix $R = q_{n,m}$ where $n, m \in \{-1, 0, 1\}$ has all its off-diagonal entries given by Proposition 4.1, and they only depend upon $\psi(1) =: a$ since $\psi(0) = 0$ and $\psi(-1) = -\psi(1) = -a$. This shows

$$\frac{\psi(n) - \psi(m)}{n - m} = a, \quad \forall n, m \in \{-1, 0, 1\}, \mid n \neq m.$$

If $a = 0$ the eigenvectors corresponding to the extremal eigenvalues of R are elements of the basis, and their Fourier transform is, by (20) a multiple of

$$\frac{\sin(s/2)}{s/2 - n\pi}, \quad n \in \{-1, 0, 1\}$$

whose zeros are all real. We can thus assume that $a = -1$. The diagonal values of R are an even function of $n \in \{-1, 0, 1\}$ and thus by adding a scalar multiple of the identity matrix one can assume that they are of the form $\{0, c, 0\}$ for some $c \in \mathbb{R}$. Let us now prove the result for the specific matrix $M(c)$ of (30). The value of $X(c)$ is always positive and thus the Fourier transform of the vector $(1, X(c), 1)$ has all its roots real for any value of c. The value of $Y(c)$ belongs to the forbidden interval $(-2, 0)$ exactly when $c < 0$, but in this case the corresponding eigenvalue $y(c)$ fails to be extremal since it is strictly between the two others $x(c)$ and 1. □

B.2. Convexity proof of Proposition B.1. We consider the positive cone C_+ in the linear space C of matrices of the form

$$\mu(a, b, c) = \begin{pmatrix} a & c & c \\ c & b & c \\ c & c & a \end{pmatrix}.$$

We first rewrite this matrix in terms of the two orthogonal projections

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

and the identity matrix 1_3. One gets

$$(b-a)P + 3cQ + (a-c)1_3 = \mu(a,b,c)$$

The ranges of the projections P, Q generate the two dimensional subspace S which is the orthogonal of the vector $v = (1, 0, -1)$ which belongs to the kernel of P and Q. The angle of the two projections P, Q is determined by its sine square,

$$(P-Q)^2 = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix} = \frac{2}{3}E, \qquad E = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

where we denote by E the orthogonal projection on E. By construction E commutes with $\mu(a,b,c)$. We take the orthonormal basis of E given by the vectors $v_1 = (0, 1, 0)$ and $v_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. In this basis the projections P, Q are given by the matrices

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix}.$$

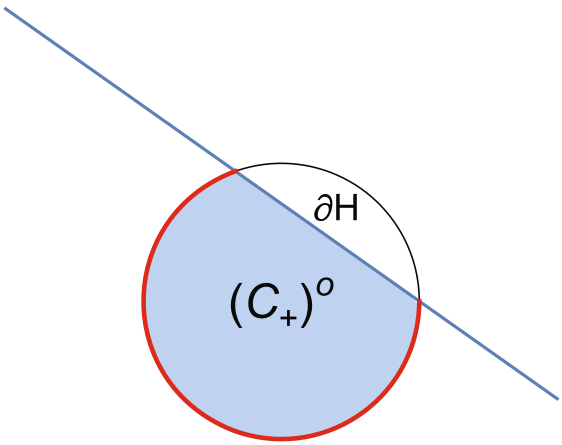
Any real symmetric 2×2 matrix can be written as

$$\begin{pmatrix} x & y \\ y & z \end{pmatrix} = \frac{1}{2}\left(2x + \sqrt{2}y - 2z\right) p + \frac{3y}{\sqrt{2}} q + (z - \sqrt{2}y)1_2.$$

Lemma B.2. (i) *The map* $\rho : C \to \mathrm{End}(E)$ *given by* $\rho(T) := ETE$ *is an isomorphism of* C *with the linear space* $S(E)$ *of selfadjoint real matrices in* E.
(ii) *The image* $\rho(C_+)$ *is the intersection of the cone* $S(E)_+$ *of positive elements of* $S(E)$, *with the half space* H,

$$H := \{\begin{pmatrix} x & y \\ y & z \end{pmatrix} \mid z \geq \sqrt{2}y\}.$$

(iii) *The extreme rays of* C_+ *are transformed by the isomorphism* ρ *into those extreme rays of* $S(E)_+$ *which are included in* H.



Proposition B.3. *The elements $T = \mu(a, b, c) \in C_+$ which have a non-zero kernel are of three kinds:*

(1) If $a > 0$, then T is on an extreme ray of C_+, its restriction to E is a positive multiple of a rank one projection whose kernel coincides with the kernel of T.
(2) If $a = 0$ and T is not on an extreme ray of C_+, the kernel of T is one dimensional, equal to $\mathbb{R}v$, $v = (1, 0, -1)$.
(3) If $a = 0$ and T is on an extreme ray of C_+, the kernel of T is 2-dimensional,

Using this proposition one obtains another proof of Proposition B.1.

B.3. The case $N = 2$. We are considering the linear space C of real symmetric matrices of the form

$$\mu(a,b,y,z,t) = \begin{pmatrix} t+z & b-a & \frac{b}{2} & \frac{a+b}{3} & \frac{b}{2} \\ b-a & t+y & a & a & \frac{a+b}{3} \\ \frac{b}{2} & a & t & a & \frac{b}{2} \\ \frac{a+b}{3} & a & a & t+y & b-a \\ \frac{b}{2} & \frac{a+b}{3} & \frac{b}{2} & b-a & t+z \end{pmatrix}.$$

The block decomposition using the subspace E for $N = 1$ works in general and corresponds to the restriction to even and odd vectors, coming from the commutativity of $\mu(a, b, y, z, t)$ with the symmetry

$$J = \begin{pmatrix} 0&0&0&0&1 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \end{pmatrix}, \qquad J^2 = \mathrm{Id}.$$

We take the orthonormal basis of even vectors, *i.e.* $E := \{\xi \mid J\xi = \xi\}$ given by

$$e_1 = (0,0,1,0,0), \quad e_2 = (0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0), \quad e_3 = (\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}})$$

and obtain the following matrix for the restriction of $\mu(a, b, y, z, t)$ to E,

$$\sigma(a,b,y,z,t) = \begin{pmatrix} t & \sqrt{2}\,a & \frac{b}{\sqrt{2}} \\ \sqrt{2}\,a & a+y+t & -\frac{2}{3}(a-2b) \\ \frac{b}{\sqrt{2}} & -\frac{2}{3}(a-2b) & \frac{b}{2}+z+t \end{pmatrix}.$$

For the odd vectors we take the orthonormal basis given by

$$n_1 = (\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}), \quad n_2 = (0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0)$$

and obtain the following matrix for the restriction of $\mu(a, b, y, z, t)$

$$\alpha(a,b,y,z,t) = \begin{pmatrix} \frac{1}{2}(2z-b)+t & -\frac{2}{3}(2a-b) \\ -\frac{2}{3}(2a-b) & -a+t+y \end{pmatrix}.$$

We now investigate positive matrices $\mu(a, b, y, z, t)$ whose kernel contains the vector $\xi = ue_1 + ve_2 + we_3 \in E$. This condition specifies a 2-dimensional subspace $K(u, v, w)$ of C of the form

$$K = \{\eta = (a, b, y, z, t) \mid y \to \frac{bw\left(3\sqrt{2}v - 8u\right)}{6uv} - \frac{a\left(3\sqrt{2}u^2 + 3uv - 2uw - 3\sqrt{2}v^2\right)}{3uv},$$

$$z \to \frac{av\left(2u + 3\sqrt{2}w\right)}{3uw} - \frac{b\left(3\sqrt{2}u^2 + 8uv + 3uw - 3\sqrt{2}w^2\right)}{6uw}, t \to -\frac{\sqrt{2}av}{u} - \frac{bw}{\sqrt{2}u}\}$$

and one needs to decide if this subspace contains a positive element of C. A 2×2 hermitian matrix is positive if and only if its two real eigenvalues are positive, and this is equivalent to the positivity of its trace and of its determinant. One applies this to the matrix $\alpha(\eta)$ and to the restriction of $\sigma(\eta)$ to the orthogonal of the vector ξ. One obtains in this manner the following 4 "positivity conditions":

(1) Trace of $\alpha \geq 0$

$$-\frac{3\sqrt{2}au + 6av - 2aw + 4bw}{3v} - \frac{-4av + 3\sqrt{2}bu + 8bv + 6bw}{6w} \geq 0$$

.

(2) Det of $\alpha \geq 0$.

$$\frac{-2a^2v\left(\sqrt{2}u + 2(v+w)\right) + ab\left(3u^2 + \sqrt{2}u(7v + 2w) + 8v^2 + 6vw - 2w^2\right) + 2b^2w\left(\sqrt{2}u + 2(v+w)\right)}{3vw} \geq 0$$

(3) Trace of $\sigma \geq 0$.

$$\frac{2a\left(-3u^2w + \sqrt{2}u\left(v^2 + w^2\right) - 3v^2w\right) - b\left(3u^2v + 4\sqrt{2}u\left(v^2 + w^2\right) + 3vw^2\right)}{3\sqrt{2}uvw} \geq 0$$

.

(4) Det of $\sigma \geq 0$.

$$-\frac{\left(u^2 + v^2 + w^2\right)\left(2\sqrt{2}a^2v + ab\left(\sqrt{2}(w - 4v) - 3u\right) - 2\sqrt{2}b^2w\right)}{3uvw} \geq 0$$

.

On the other hand the polynomial associated to $\xi = ue_1 + ve_2 + we_3 \in E$ is

$$P(s) = s^4u + \sqrt{2}s^4v + \sqrt{2}s^4w - 20\pi^2 s^2u - 16\sqrt{2}\pi^2 s^2v - 4\sqrt{2}\pi^2 s^2w + 64\pi^4u$$

which depends only on s^2 and has real roots when the degree 2 polynomial

$$ux^2 - 20\pi^2ux + 64\pi^4u + \sqrt{2}vx^2 - 16\sqrt{2}\pi^2vx + \sqrt{2}wx^2 - 4\sqrt{2}\pi^2wx$$

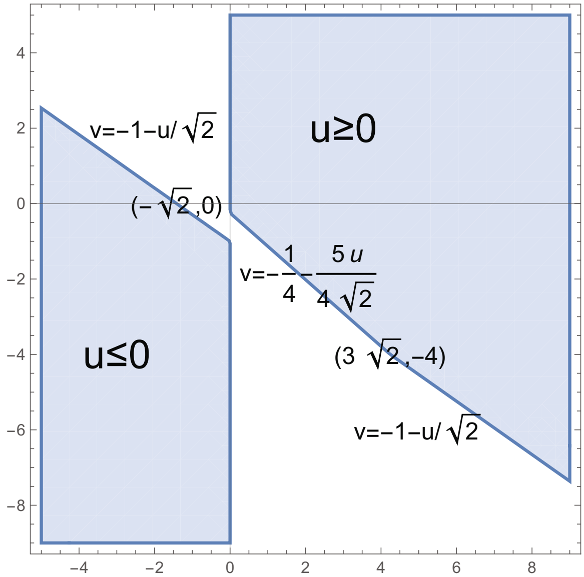
has positive roots, *i.e.* equivalently that the sum and product of the roots are positive, which gives the two conditions[2]

$$\frac{5u + \sqrt{2}(4v + w)}{u} > 0, \quad \frac{u + \sqrt{2}v + \sqrt{2}w}{u} > 0. \tag{31}$$

[2] We use the reciprocal polynomial.

In fact we can assume that $w \neq 0$ since otherwise one is reduced to the case $n = 1$. Thus we take $w = 1$, and (31) is reduced to the following cases :

$$\left(u < 0 \wedge v < \frac{-u-\sqrt{2}}{\sqrt{2}}\right) \vee \left(0 < u < 3\sqrt{2} \wedge v > \frac{-5u-\sqrt{2}}{4\sqrt{2}}\right)$$
$$\vee \left(u \geq 3\sqrt{2} \wedge v > \frac{-u-\sqrt{2}}{\sqrt{2}}\right). \tag{32}$$



This shows the region of the (u, v) plane near the origin which ensures that the zeros of the associated polynomial P are real.
We now reduce the above 4 positivity conditions, and compare them with (32).

$u \leq -\sqrt{2}$ In this case the reduction of the above 4 positivity conditions gives

$$\left(v < 0 \vee 0 < v < \frac{1}{2}\left(\sqrt{2}(-u)-2\right)\right) \Big\| u = -\sqrt{2} \wedge v < 0$$

which implies (32).

$-\sqrt{2} < u < 0$ In this case the reduction of the above 4 positivity conditions gives

$$-\sqrt{2} < u < 0 \wedge v < \frac{1}{2}\left(\sqrt{2}(-u)-2\right)$$

which implies (32).

$0 < u < 3\sqrt{2}$ In this case the reduction of the above 4 positivity conditions gives

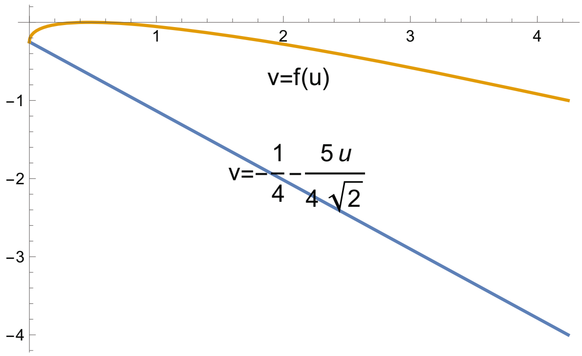
$$\left(0 < u < \frac{\sqrt{2}}{3} \wedge \left(\frac{1}{8}\left(-3\sqrt{2}u - 2\right) + \frac{\sqrt{3}\sqrt{u}}{2\sqrt[4]{2}} < v < 0 \vee v > 0\right)\right) \vee \left(u = \frac{\sqrt{2}}{3} \wedge v > 0\right)$$

$$\vee \left(\frac{\sqrt{2}}{3} < u \leq 3\sqrt{2} \wedge \left(\frac{1}{8}\left(-3\sqrt{2}u - 2\right) + \frac{\sqrt{3}\sqrt{u}}{2\sqrt[4]{2}} < v < 0 \vee v > 0\right)\right)$$

and the solutions form the upper graph of the function

$$f(u) = \frac{1}{8}\left(-3\sqrt{2}u - 2\right) + \frac{\sqrt{3}\sqrt{u}}{2\sqrt[4]{2}}.$$

Thus to compare with (32) we need to see if the graph of f in the interval $[0, 3\sqrt{2}]$ is above the graph of the function $v = -\frac{5u}{4\sqrt{2}} - \frac{1}{4}$. This is shown by the following



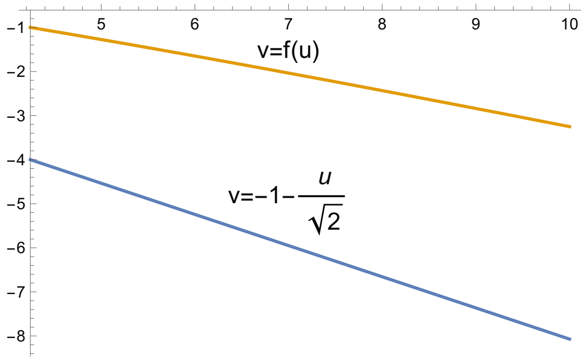
$u > 3\sqrt{2}$ In this case the reduction of the above 4 positivity conditions gives

$$\left(\frac{1}{2}\left(\sqrt{2}(-u) - 2\right) < v < \frac{1}{8}\left(-3\sqrt{2}u - 2\right) - \frac{\sqrt{3}\sqrt{u}}{2\sqrt[4]{2}}\right.$$
$$\left. \vee \frac{1}{8}\left(-3\sqrt{2}u - 2\right) + \frac{\sqrt{3}\sqrt{u}}{2\sqrt[4]{2}} < v < 0 \vee v > 0\right).$$

In the first part, the condition $-1 - u/\sqrt{2} < v$ implies (32). The second condition means that one is above the graph of the function $f(u)$. Thus we need to show that for $u > 3\sqrt{2}$ the graph of f is above the graph of $-1 - u/\sqrt{2}$. This follows from the inequality of the slopes

$$-3\sqrt{2}/8 > -1/\sqrt{2}$$

and the following



We can thus conclude that if an even vector $\xi = ue_1 + ve_2 + we_3 \in E$ is in the kernel of a positive matrix $\mu(a, b, y, z, t)$ then the zeros of the associated polynomial are real. But we need to look at the possibility of a non trivial odd vector in the kernel of a positive matrix $\mu(a, b, y, z, t)$.
We now consider the case where an odd vector $\eta = un_1 + vn_2$ is in the kernel of a positive matrix $\mu(a, b, y, z, t)$. In this case the restriction to the odd part will be a multiple of a one dimensional projection P_1 and we thus need to first solve the equation

$$\alpha(a, b, y, z, t) = \begin{pmatrix} \cos^2(\beta) & \sin(\beta)\cos(\beta) \\ \sin(\beta)\cos(\beta) & \sin^2(\beta) \end{pmatrix}$$

The solution is given by

$$\Big\{b \to 2a + \frac{3}{2}\sin(\beta)\cos(\beta),\ z \to \frac{1}{4}\left(-4\sin^2(\beta) + 4\cos^2(\beta) + 3\sin(\beta)\cos(\beta)\right) \\ +y,\ t \to a + \sin^2(\beta) - y\Big\}$$

We then consider the restriction of the solution matrix to the even part

$$\begin{pmatrix} a + \sin^2(\beta) - y & \sqrt{2}a & \frac{2a+\frac{3}{4}\sin(2\beta)}{\sqrt{2}} \\ \sqrt{2}a & 2a + \sin^2(\beta) & 2a + \sin(2\beta) \\ \frac{2a+\frac{3}{4}\sin(2\beta)}{\sqrt{2}} & 2a + \sin(2\beta) & 2a + \frac{3}{4}\sin(2\beta) + \cos^2(\beta) \end{pmatrix}$$

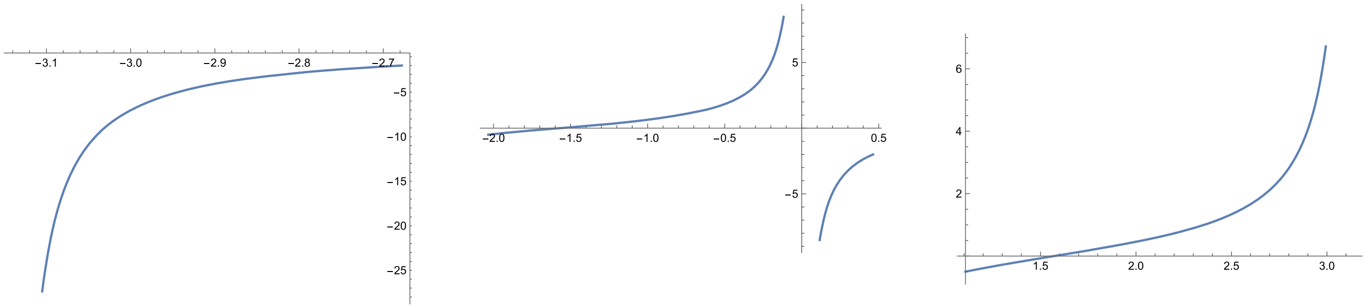
and compute its characteristic polynomial. We then apply the following

Fact B.4. *Let $P(x) = x^n + \sum a_j x^{n-j}$ be a monic polynomial whose all roots are real. Then all the roots are ≥ 0 if and only if $(-1)^j a_j \geq 0$ for all $j \in \{1, \ldots, n\}$.*

One obtains in this manner the following 3 "positivity conditions":

(1) $-a_3 \geq 0$

$$\frac{1}{8}\left(4a\sin(\beta) + \cos(\beta)\left(3\sin^2(\beta) - 2a\right)\right) \\ \left(4\sin^3(\beta) - 4y\sin(\beta) + \cos(\beta)\left(8y - 11\sin^2(\beta)\right)\right) \geq 0,$$



$$-\pi < \beta < -2\,\mathrm{ArcTan}\left(\sqrt{5}+2\right)$$

$$2\,\mathrm{ArcTan}\left(\frac{-\sqrt{5}-1}{2}\right) < \beta < -2\,\mathrm{ArcTan}\left(2-\sqrt{5}\right)$$

$$2\,\mathrm{ArcTan}\left(\frac{\sqrt{5}-1}{2}\right) < \beta \leq \pi$$

Fig. 3. Three conditions on β illustrated with corresponding graphs

(2) $a_2 \geq 0$

$$-\frac{13}{4}a\sin(2\beta) - \left(2a + \frac{1}{2}\right)\cos(2\beta) - 4ay + 5a + \frac{3}{4}\sin(2\beta) - \frac{3}{8}\sin(4\beta)$$
$$+\frac{33}{64}\cos(4\beta) - \frac{3}{4}y\sin(2\beta) - y - \frac{1}{64} \geq 0,$$

(3) $-a_1 \geq 0$

$$5a + 2\sin^2(\beta) + \cos^2(\beta) + \frac{3}{2}\sin(\beta)\cos(\beta) - y \geq 0$$

.

The solution of the existence of (a, y) fulfilling these inequalities for a given β is given by the following three cases. In each of them we plot the value of $-\cot(\beta)$ which gives the component v of the vector $n_1 + vn_2$ in the kernel of the positive matrix $\mu(a, b, y, z, t)$. We find that all values of v arise except those in the interval $(-2, -\frac{1}{2})$. The associated polynomial to the vector $un_1 + vn_2$ is

$$Q(s) = 2\sqrt{2}\left(-2\pi s^2 u - \pi s^2 v + 8\pi^3 u + 16\pi^3 v\right)$$

and its roots are real if and only if

$$\frac{8\pi^2(u + 2v)}{2u + v} \geq 0$$

which for $u = 1$ is realized if and only if $v \notin (-2, -\frac{1}{2})$ as shown by the graph of the function $\frac{2v+1}{v+2}$.
Finally the case by case discussion of the allowed values of β is given in Fig. 3.

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