

# A generalization of K-theory to operator systems

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## Chapter One.

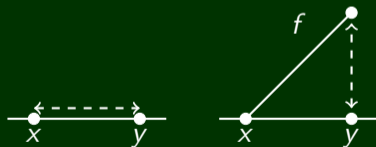
### Motivation

# Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

- ▶ Distance  $d(x, y)$  between two points is usually defined as *the **smallest** of the arclengths (computed using the metric) of curves connecting  $x$  and  $y$ .*
- ▶ But it can also be defined as *the **largest** of differences  $|f(x) - f(y)|$  for functions  $f$  with gradient  $|\nabla f| \leq 1$ .*

$$d(x, y) = \sup_{\| [D_M, f] \| \leq 1} |\delta_x(f) - \delta_y(f)|$$



Combination  $(C^\infty(M), L^2(S_M), D_M)$   
allows for reconstruction of geometry

# Spectral triples

More generally, we consider a triple  $(\mathcal{A}, \mathcal{H}, D)$

- ▶ a unital  $*$ -algebra  $\mathcal{A}$
- ▶ a self-adjoint operator  $D$  with compact resolvent and bounded commutators  $[D, a]$  for  $a \in \mathcal{A}$
- ▶ both acting (boundedly, resp. unboundedly) on Hilbert space  $\mathcal{H}$

Generalized distance function:

- ▶ States are positive linear functionals  $\phi : \mathcal{A} \rightarrow \mathbb{C}$  of norm 1
- ▶ Distance function on state space  $S(\mathcal{A})$  of  $\mathcal{A}$ :

$$d_D(\phi, \psi) = \sup_{a \in \mathcal{A}} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

These notions continue to make sense when we replace  $\mathcal{A}$  by any self-adjoint vector space  $\mathcal{E}$  of bounded operators on  $\mathcal{H}$  that contains the unit, a so-called *operator system*.

## Spectral data: $(\mathcal{A}, \mathcal{H}, D)$

- ▶ The mathematical reformulation of geometry in terms of spectral data (global analysis) requires the knowledge of the full Dirac operator.
- ▶ From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.
- ▶ Also computer simulations are inevitably finite-dimensional, but still index computations can be done (spectral localizer).

Mathematically, this is realized by **spectral truncations** of  $(A, \mathcal{H}, D)$  :

- ▶  $\mathcal{H} \mapsto P\mathcal{H}$ , spectral projection onto closed Hilbert subspace
- ▶  $D \mapsto PDP$ , still a self-adjoint operator
- ▶  $A \mapsto PAP$ , this is not an algebra any more (unless  $P \in A$ )

Instead,  $PAP$  is an operator system:  $(PaP)^* = Pa^*P$

## Chapter Two.

### Operator Systems

# Abstract operator systems

## Definition

We say that a  $*$ -vector space is matrix ordered if

1. for each  $n$  we are given a cone of positive elements  $M_n(E)_+$  in  $M_n(E)_h$ ,
2.  $M_n(E)_+ \cap (-M_n(E)_+) = \{0\}$  for all  $n$ ,
3. for every  $m, n$  and  $A \in M_{mn}(\mathbb{C})$  we have that  $AM_n(E)_+A^* \subseteq M_m(E)_+$ .

We call  $e \in E_h$  an order unit for  $E$  if for each  $x \in E_h$  there is a  $t > 0$  such that  $-te \leq x \leq te$ . It is called an Archimedean order unit if  $-te \leq x$  for all  $t > 0$  implies that  $x \geq 0$ .

## Definition

An (abstract) operator system is given by a matrix-ordered  $*$ -vector space  $E$  with an order unit  $e$  such that for all  $n$   $e^{\oplus n}$  is an Archimedean order unit for  $M_n(E)$ .

Maps between operator systems  $E, F$  are completely positive maps in the sense that their extensions  $M_n(E) \rightarrow M_n(F)$  are positive for all  $n$ .

Isomorphisms are complete order isomorphisms

# $C^*$ -envelope of a unital operator system

[Arveson, 1969]

Hamana: existence and uniqueness in 1979; realized á la Arveson as direct sum of all boundary representations [Dritschel–McCullough 2005, Arveson 2008, Davidson–Kennedy 2015]

A  $C^*$ -extension  $\kappa : E \rightarrow A$  of a unital operator system  $E$  is given by a complete order isomorphism onto  $\kappa(E) \subseteq A$  such that  $C^*(\kappa(E)) = A$ .

A  $C^*$ -envelope of a unital operator system is a  $C^*$ -extension  $\kappa : E \rightarrow A$  with the following universal property:

$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & \searrow \lambda & \uparrow \exists! \rho \\ & & B \end{array}$$

Example: operator system  $C_{\text{harm}}(\overline{\mathbb{D}})$  of continuous harmonic functions with  $C^*$ -envelope  $C(S^1)$ .

## How far is an operator system from a $C^*$ -algebra?

One lets  $E^{\circ n}$  be the norm closure of the linear span of products of  $\leq n$  elements of  $E$ .

### Definition

*The propagation number  $\text{prop}(E)$  of  $E$  is defined as the smallest integer  $n$  such that  $(\iota_E(E))^{\circ n} \subseteq C_{\text{env}}^*(E)$  is a  $C^*$ -algebra.*

Returning to the harmonic functions in the disk we have  $\text{prop}(C_{\text{harm}}(\overline{\mathbb{D}})) = 1$ .

### Proposition (Connes-vS, 2020; Pawłowska, 2024)

*The propagation number is invariant under complete order isomorphisms, as well as under stable=Morita equivalence [EKT, 2019]:*

$$\text{prop}(E) = \text{prop}(E \otimes_{\min} \mathcal{K})$$

*More generally [Koot, 2021], we have*

$$\text{prop}(E \otimes_{\min} F) = \max\{\text{prop}(E), \text{prop}(F)\}$$

Chapter Three.

$S^1$

## Example: spectral truncation of the circle [Connes-vS, 2020]

- ▶ Eigenvectors of  $D_{S^1}$  are Fourier modes  $e_k(t) = e^{ikt}$  for  $k \in \mathbb{Z}$
- ▶ Orthogonal projection  $P = P_n$  onto  $\text{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_n\}$
- ▶ The space  $C(S^1)^{(n)} := PC(S^1)P$  is the operator system of Toeplitz matrices:

$$PfP \sim (t_{k-l})_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ & & & \ddots & \ddots \\ t_{n-2} & & & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

- ▶ States are defined as unital positive linear functionals.

We have:  $C_{\text{env}}^*(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$  and  $\text{prop}(C(S^1)^{(n)}) = 2$  (for any  $n$ ).

# Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system  $C^*(\mathbb{Z})_{(n)}$ :

- ▶ functions on  $S^1$  with a finite number of non-zero Fourier coefficients:

$$a = (\dots, 0, a_{-n+1}, a_{-n+2}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-2}, a_{n-1}, 0, \dots)$$

- ▶ an element  $a$  is positive iff  $\sum_k a_k e^{ikx}$  is a positive function on  $S^1$ .
- ▶ The  $C^*$ -envelope of  $C^*(\mathbb{Z})_{(n)}$  is given by  $C^*(\mathbb{Z}) \cong C(S^1)$ ; propagation number  $\infty$

## Proposition

1. *The extreme rays in  $(C^*(\mathbb{Z})_{(n)})_+$  are given by the elements  $a = (a_k)$  for which the Laurent series  $\sum_k a_k z^k$  has all its zeroes on  $S^1$ .*
2. *The pure states of  $C^*(\mathbb{Z})_{(n)}$  are given by  $a \mapsto \sum_k a_k \lambda^k$  ( $\lambda \in S^1$ ).*

# Pure states on the Toeplitz matrices

Duality of  $C(S^1)^{(n)}$  and  $C^*(\mathbb{Z})_{(n)}$  [Connes–vS 2020] and [Farenick 2021]:

$$\begin{aligned} C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} &\rightarrow \mathbb{C} \\ (T = (t_{k-l})_{k,l}, a = (a_k)) &\mapsto \sum_k a_k t_{-k} \end{aligned}$$

## Proposition

1. The extreme rays in  $C(S^1)_+^{(n)}$  are  $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$  for any  $\lambda \in S^1$ .
2. The pure state space  $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$ .

# Curiosities on Toeplitz matrices

Theorem (Carathéodory)

Let  $T$  be an  $n \times n$  Toeplitz matrix. Then  $T \geq 0$  iff  $T = V\Delta V^*$  with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \dots & \\ & & & d_n \end{pmatrix}; \quad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix},$$

for some  $d_1, \dots, d_n \geq 0$  and  $\lambda_1, \dots, \lambda_n \in S^1$ .

Chapter Five.

K-theory

# K-theory for operator systems

[arXiv:2409.02773]

A key invariant of  $C^*$ -algebras is K-theory. Is there an analogue for operator systems?

- ▶ Need notion of projection (*cf.* Araiza–Russell) or invertible selfadjoint elements
- ▶ It should capture the spectral localizer of Loring, Schulz-Baldes, and others
- ▶ It should be invariant under Morita equivalence [EKT]

## Definition

*A hermitian form  $x$  in a unital operator system  $E$  is a selfadjoint element  $x \in M_n(E)$  which is non-degenerate in the sense that there exists  $g > 0$  such that for all pure and maximal ucp maps  $\phi : E \rightarrow B(\mathcal{H})$  we have*

$$|\phi^{(n)}(x)| \geq g \cdot \text{id}_{\mathcal{H}}^{\oplus n}$$

*In other words,  $x$  should have a gap  $g$  in each boundary representation*

We will write  $H(E, n)$  for all hermitian forms in  $M_n(E)$ .

### Proposition

*An element  $x \in M_n(E)$  is non-degenerate if and only if  $\iota_E^{(n)}(x)$  is an invertible element in the  $C^*$ -envelope  $C_{env}^*(E)$ .*

This is a consequence of the realization of the  $C^*$ -envelope in [Davidson–Kennedy]

### Examples:

1. Hermitian forms (à la Witt) on a fgp right module  $pA^n$  over a  $C^*$ -algebra  $A$ : described by invertible elements  $x = h + (1 - p) \in M_n(A)$  with  $h \in pM_n(A)p$ .
2. Projections  $p$  in operator systems à la Araiza–Russell are precisely projections in the  $C^*$ -envelope:  $x = e - 2p$  is a hermitian form.
3. Similarly,  $\epsilon$ -projections in quantitative K-theory [Oyono-Oyono–Yu] define hermitian forms.
4. Spectral compressions of projections in  $C^*$ -algebra:  $x = PYP$  with  $Y = 1 - 2p$  provided  $\|[P, p]\|$  sufficiently small.

# The invariants and K-theory

$$\mathcal{V}(E, n) = H(E, n) / \sim_n$$

Example:

$$\mathcal{V}(\mathbb{C}, n) \cong \{-n, -n+2, \dots, n\}$$

and with the map  $\iota_{nm}([x] = x \oplus e_{m-n}$  we have

$$\begin{array}{ccc} \mathcal{V}(\mathbb{C}, n) & \xrightarrow{\iota_{nm}} & \mathcal{V}(\mathbb{C}, m) \\ & \searrow \rho_n & \swarrow \rho_m \\ & \mathbb{N} & \end{array}$$

In general, we consider

$$\mathcal{V}(E) = \varinjlim \mathcal{V}(E, n)$$

and  $K_0(E)$  is the corresponding Grothendieck group (with identity  $[e]$  and addition  $'\oplus'$ )

# Properties of $K_0$

- ▶ For  $C^*$ -algebras we obtain usual K-theory via the map  $[x] \mapsto [p = \frac{1}{2}(1 - x|x|^{-1})]$ .
- ▶ Stability: we define a map  $\iota_n : M_n(E) \rightarrow M_n(M_2(E))$  by

$$\iota_n(x) = \begin{pmatrix} x_{11} & 0 & x_{12} & 0 & \cdots & x_{1n} & 0 \\ 0 & e & 0 & 0 & \cdots & 0 & 0 \\ \hline x_{21} & 0 & x_{22} & 0 & \cdots & & \vdots \\ 0 & 0 & 0 & e & \cdots & & \vdots \\ \hline \vdots & & \vdots & & \ddots & & \vdots \\ \hline x_{n1} & 0 & & & \cdots & x_{nn} & 0 \\ 0 & 0 & & \cdots & \cdots & 0 & e \end{pmatrix}$$

so that  $\iota_n(x) \sim x$  (Whitehead). This allows to show  $K_0(E) \cong K_0(M_2(E))$ .

# Non-unital operator systems and stability

The unitization [Werner, 2002] of a non-unital operator system  $E$  is given by the  $*$ -vector space  $E^+ = E \oplus \mathbb{C}$  with matrix order structure:

$$(x, A) \geq 0 \text{ iff } A \geq 0 \text{ and } \phi(A_\epsilon^{-1/2} x A_\epsilon^{-1/2}) \geq -1$$

for all  $\epsilon > 0$  and noncommutative states  $\phi \in \mathcal{S}_n(E)$ , and where  $A_\epsilon = \epsilon \mathbb{I}_n + A$ .

$$\tilde{\mathcal{V}}(E, n) := \{(x, A) \in H(E^+, n) : A \sim_n \mathbb{I}_n\} / \sim_n$$

In the unital case, the isomorphism  $E^+ \cong E \oplus \mathbb{C}$  given by  $(x, A) \mapsto (x + Ae, A)$  yields that in this case

$$\tilde{\mathcal{V}}(E, n) \cong \mathcal{V}(E, n).$$

Theorem

*For a unital operator system  $E$  we have  $K_0(\mathcal{K} \otimes E) \cong K_0(E)$ .*

# Stability

## Theorem

For a unital operator system  $E$  we have  $K_0(\mathcal{K} \otimes E) \cong K_0(E)$ .

Proof.

1. Realize stabilization by maps  $\kappa_{NM} : M_N(E) \rightarrow M_M(E)$ ,  $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0_{M-N} \end{pmatrix}$
2. Commuting diagram:

$$\begin{array}{ccc} \tilde{\mathcal{V}}(E, n) & \xrightarrow{\kappa_{1N}} & \tilde{\mathcal{V}}(M_N(E), n) \\ \cong \downarrow & & \cong \downarrow \\ \mathcal{V}(E, n) & \xrightarrow[\cong]{\iota_n} & \mathcal{V}(M_N(E), n) \end{array}$$

3. The map  $\kappa_{1\infty} : K_0(E) \rightarrow K_0(\mathcal{K} \otimes E)$  is an isomorphism:
  - ▶ injective: homotopy in  $H((\mathcal{K} \otimes E)^+, n)$  compressed to homotopy in  $H((M_N(E))^+, n)$ .
  - ▶ surjective: approximation by finite-rank operators in norm is still hermitian form.



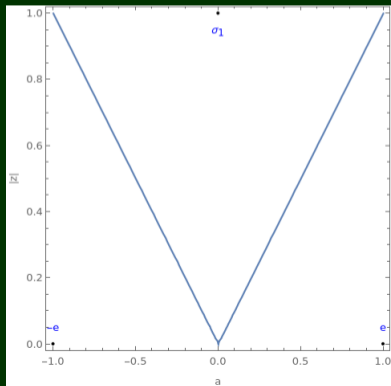
Chapter Six.

Applications

## Example: Toeplitz matrices

- ▶ Consider the operator system  $C(S^1)^{(2)}$  of  $2 \times 2$  Toeplitz matrices.
- ▶ Hermitian forms in  $H(C(S^1)^{(2)}, 1)$  are matrices of the form

$$T = \begin{pmatrix} a & z \\ \bar{z} & a \end{pmatrix}; \quad a^2 - |z| \neq 0.$$



- ▶  $\mathcal{V}(C(S^1)^{(2)}, 1) \cong \{[-e], [\sigma_1], [e]\}$
- ▶ However,  $\sigma_1 \oplus \sigma_1 \sim e \oplus (-e)$  in  $H(C(S^1)^{(2)}, 2)$ :

$$h(t) = \begin{pmatrix} (1-t)\sigma_1 + te & it(t-1)\sigma_2 \\ -it(t-1)\sigma_2 & (1-t)\sigma_1 - te \end{pmatrix}$$

with  $\det h(t) > 0$ .

## Example: spectral localizer on the 2-sphere

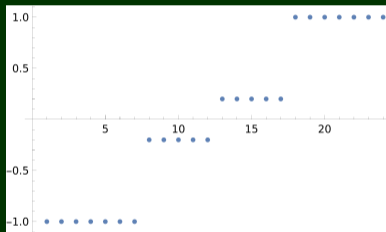
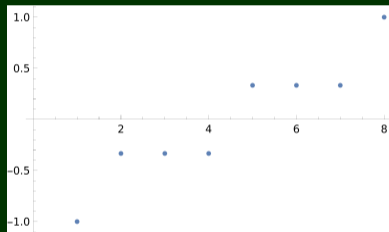
- ▶ Consider the Bott projection:

$$p = \frac{1}{2} \begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix} \in M_2(C(S^2))$$

for  $x^2 + y^2 + z^2 = 1$ .

- ▶ Dirac operator  $D_{S^2} \rightsquigarrow$  spinorial harmonics  $Y_{lm}^\pm$  [GVF] with eigenvalues  $\pm(l + 1/2)$  ( $l = 1/2, 3/2, \dots$ ).
- ▶ Spectral projections  $P_\rho$  onto  $\text{span}_{\mathbb{C}}\{Y_{lm}^\pm\}_{l \leq \rho} \subset L^2(\mathcal{S}_{S^2})$ .
- ▶ We obtain a compression  $P_\rho Y P_\rho$  of the hermitian form  $Y = 1 - 2p$  on  $\mathbb{T}^2$  corresponding to  $p$ .

Eigenvalues for  $P_\rho Y P_\rho$  (Bott) for  $\rho = 1/2, 3/2, 5/2$



# Spectral localizer

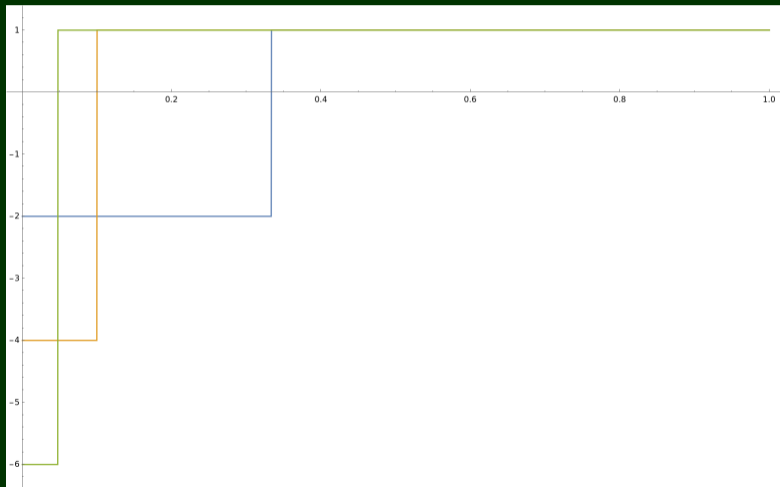
- ▶ The *spectral localizer* of Loring and Schulz-Baldes is given by the following matrix:

$$L_{\kappa,\rho} = \begin{pmatrix} -PYP & \kappa PD^+P \\ \kappa PD^-P & PYP \end{pmatrix}$$

In general they show that for suitable  $\kappa$  and  $\rho$  the index pairing can be computed as the signature of this matrix:

$$\text{Index } \rho D^+ \rho = \frac{1}{2} \text{Sig } L_{\kappa,\rho}$$

Signature of spectral localizer  $L_{\kappa,\rho}$  for Bott (as a function of  $\kappa$  for  $\rho = 1/2, 3/2, 5/2$ )



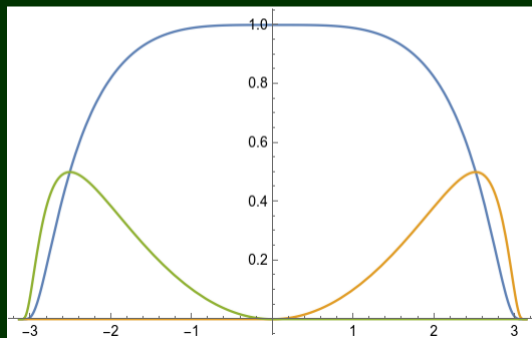
## Example: spectral localizer on the 2-torus

Loring considered (in his thesis!) analogues of the Powers–Rieffel projections:

$$p = \begin{pmatrix} f & g + hU \\ g + hU^* & 1 - f \end{pmatrix} \in M_2(C^\infty(\mathbb{T}^2))$$

with  $U$  a unitary in the second variable, and  $f, g, h$  real-valued smooth functions in the first variable, satisfying

$$gh = 0, \quad g^2 + h^2 = f - f^2.$$



## Spectral truncations on $\mathbb{T}^2$

- ▶ We now consider spectral truncations  $P = P_\rho$  onto  $\ell^2\{\vec{n} \in \mathbb{Z}^2 : \|\vec{n}\| \leq \rho\} \subseteq \ell^2(\mathbb{Z}^2)$ .
- ▶ We obtain a compression  $PYP$  of the hermitian form  $Y = 1 - 2\rho$  on  $\mathbb{T}^2$  corresponding to  $\rho$ :

$$PYP = \begin{pmatrix} P - 2PfP & -2PgP - 2PhUP \\ -2PgP - 2PhU^*P & -P + 2PfP \end{pmatrix} \in M_2(PC^\infty(\mathbb{T}^2)P)$$

For suitable  $P$  these are hermitian forms  $\rightsquigarrow [PYP] \in K_0(PC(\mathbb{T}^2)P)$ .

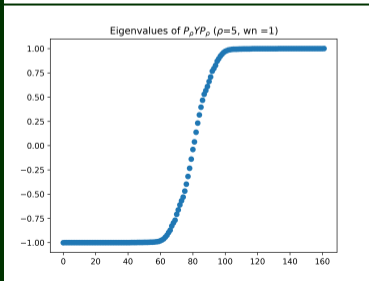
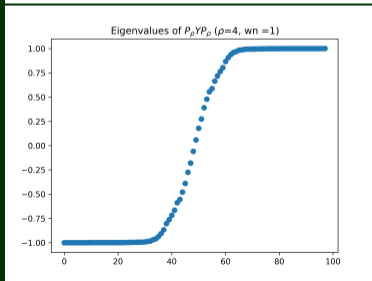
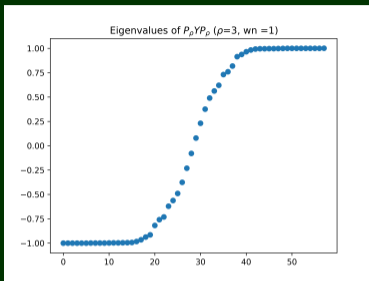
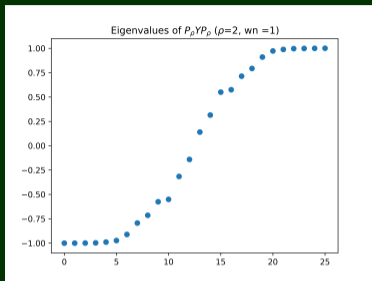
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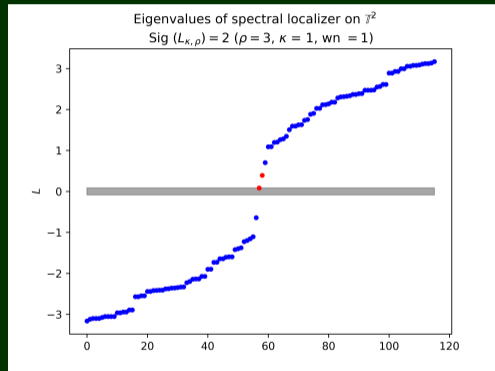
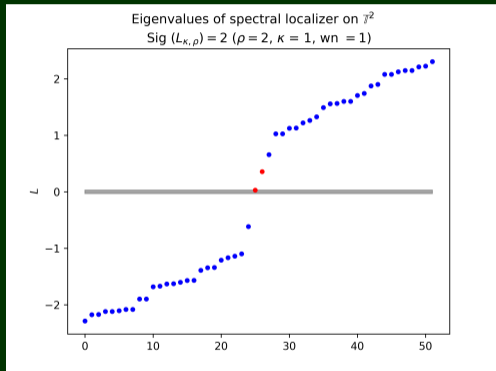
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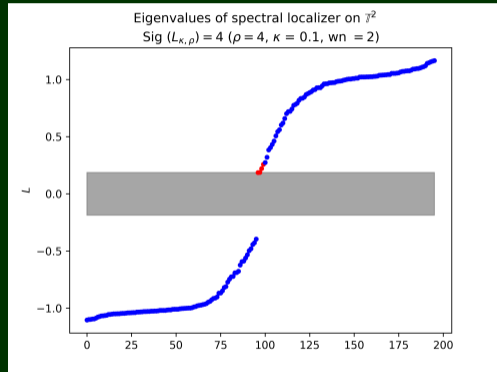
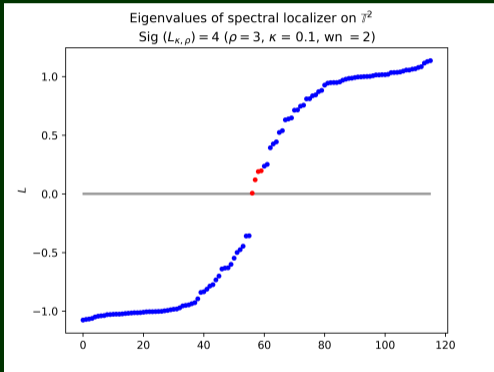
# Simulations: eigenvalues of $PYP$ for $U(t_2) = e^{it_2}$



# Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{it_2}$



# Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{2it_2}$



# Summary and outlook

- ▶ Noncommutative geometry: metric aspect
- ▶ Operator systems: from (spectral) truncations
- ▶ Duality of operator systems: state spaces
- ▶ New invariants: propagation number, K-theory
  - ▶ Higher K-group invariants [arXiv:2411.02981]:

$$\mathcal{V}_1^\delta(E, n) = \left\{ x \in M_n(E) : \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \text{ has spectral gap } \delta \right\} / \sim_n$$

and, more generally,  $\mathcal{V}_p^\delta(E, n) := H^\delta(E \otimes \mathbb{C}l_p^{(1)}, n) / \sim_n$ .

- ▶ Formal periodicity:  $K_{2m}(E) = K_0(E)$  and  $K_{2m+1}(E) = K_1(E)$ .

- ▶ Persistence for projective systems:

$$A \rightarrow \cdots \rightarrow E_k \rightarrow E_{k+1} \rightarrow \cdots$$

- ▶ When does  $[H] \in K_0(A)$  induce a (non-trivial) class in  $K_0(E_k)$  for some  $k$ ?
  - ▶ Can use spectral localizer (*cf.* 2-torus) to check non-triviality.
- ▶ Persistence for inductive systems:

$$\cdots \rightarrow E_{k-1} \rightarrow E_k \rightarrow \cdots \rightarrow A$$

- ▶ When do we have that  $[x] \in K_0(E_k)$  persists to induce a (non-trivial) class in  $K_0(A)$ ?
  - ▶ Can we extract invariants of  $A$  from the invariants  $\{\mathcal{V}(E_k, k)\}_k$ ?
  - ▶ Relation to quantitative K-theory [Oyono-Oyono–Yu] and NF/CPC\*-systems [Blackadar–Kirchberg, Courtney–Winter]