A generalization of K-theory to operator systems

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Chapter One.

Motivation

Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

- Distance $d(x, y)$ between two points is usually defined as

  the **smallest** *of the arclengths (computed using the metric) of curves connecting x and y.*

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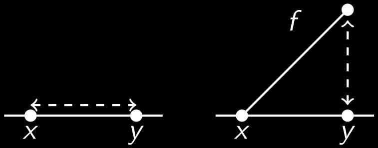

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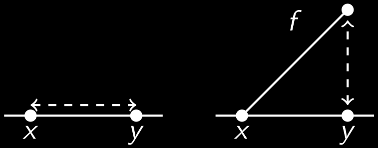

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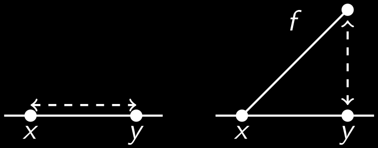

Combination $(C^\infty(M), L^2(S_M), D_M)$
allows for reconstruction of geometry

Spectral triples

More generally, we consider a triple $(\mathcal{A}, \mathcal{H}, D)$

- a unital $*$-algebra $\mathcal{A}$
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A}$
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Generalized distance function:

- States are positive linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ of norm 1
- Distance function on state space $S(\mathcal{A})$ of $\mathcal{A}$:

$$d_D(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$

These notions continue to make sense when we replace $\mathcal{A}$ by any self-adjoint vector space $\mathcal{E}$ of bounded operators on $\mathcal{H}$ that contains the unit, a so-called *operator system*.

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Mathematically, this is realized by **spectral truncations** of $(A, \mathcal{H}, D)$:

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Instead, PAP is an operator system: $(PaP)^* = Pa^*P$

Chapter Two.

Operator
Systems

Abstract operator systems

Definition

We say that a $*$-vector space is matrix ordered *if*

1. *for each n we are given a cone of positive elements $M_n(E)_+$ in $M_n(E)_h$,*
2. $M_n(E)_+ \cap (-M_n(E)_+) = \{0\}$ *for all n,*
3. *for every m, n and $A \in M_{mn}(\mathbb{C})$ we have that $AM_n(E)_+A^* \subseteq M_m(E)_+$.*

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We call $e \in E_h$ an *order unit* for E if for each $x \in E_h$ there is a $t > 0$ such that $-te \leq x \leq te$. It is called an *Archimedean order unit* if $-te \leq x$ for all $t > 0$ implies that $x \geq 0$.

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Maps between operator systems E, F are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

Isomorphisms are complete order isomorphisms

C^*-envelope of a unital operator system

[Arveson, 1969]

Hamana: existence and uniqueness in 1979; realized á la Arveson as direct sum of all boundary representations [Dritschel–McCullough 2005, Arveson 2008, Davidson–Kennedy 2015]

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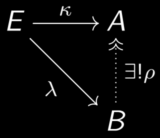

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$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & {\scriptstyle \lambda}\searrow & \uparrow {\scriptstyle \exists! \rho} \\ & & B \end{array}$$

Example: operator system $C_{\text{harm}}(\overline{\mathbb{D}})$ of continuous harmonic functions with C^*-envelope $C(S^1)$.

How far is an operator system from a C^*-algebra?

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E.

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The propagation number $\text{prop}(E)$ *of* E *is defined as the smallest integer* n *such that* $(\imath_E(E))^{\circ n} \subseteq C^*_{env}(E)$ *is a* C^**-algebra.*

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Proposition (Connes-vS, 2020; Pawlowska, 2024)

The propagation number is invariant under complete order isomorphisms, as well as under stable=Morita equivalence [EKT, 2019]:

$$prop(E) = prop(E \otimes_{min} \mathcal{K})$$

More generally [Koot, 2021], we have

$$prop(E \otimes_{\min} F) = \max\{prop(E), prop(F)\}$$

Chapter Three.

S^1

Example: spectral truncation of the circle [Connes-vS, 2020]

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is the operator system of Toeplitz matrices:

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

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We have: $C^*_{\mathrm{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\mathrm{prop}(C(S^1)^{(n)}) = 2$ (for any n).

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

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Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
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Proposition

1. *The extreme rays in* $C(S^1)_+^{(n)}$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$.

Curiosities on Toeplitz matrices

Theorem (Carathéodory)

Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ iff $T = V \Delta V^*$ with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \qquad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \dots, d_n \geq 0$ and $\lambda_1, \dots, \lambda_n \in S^1$.

Chapter Four.

Bonds

Operator systems, groupoids and bonds (aka a positivity domain)

Definition (Connes-vS, 2021)

A bond is a triple (G, ν, Ω) *consisting of a locally compact groupoid* G*, a Haar system* $\nu = \{\nu_x\}$ *and an open symmetric subset* $\Omega \subseteq G$ *containing the units* $G^{(0)}$*.*

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Proposition

Let (Ω, G, ν) *be a bond. The closure of the subspace* $C_c(\Omega) \subseteq C_c(G)$ *in the* C^**-algebra* $C^*(G)$ *is a (possibly non-unital) operator system.*

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1. *Consider* Ω *in a l.c. Lie group* $G \rightsquigarrow$ *Fourier truncations (á la Rieffel) in* $C^*(G)$
2. *Consider* $\Omega_n = (-n, n) \subset \mathbb{Z} \rightsquigarrow$ *Fejér–Riesz system* $C^*(\mathbb{Z})_{(n)} \cong (C(S^1)^{(n)})^d$*.*
3. *Consider* $\Omega_n = (-n, n) \subseteq C_m$ *(so modulo* m*). The operator system consists of banded* $m \times m$ *circulant matrices of band width* n*.*

Tolerance relations on finite sets [Gielen–vS, 2022]

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2. If $\mathcal{R}$ is a chordal graph, then $E(\mathcal{R})^d \cong E(\mathcal{R})$ as a vector space, but with order structure given by being partially positive.
3. Concrete realization of $E(\mathcal{R})^d$ in terms of cliques C in $\mathcal{R}$:

$$\Phi : E(\mathcal{R})^d \to \bigoplus_{C \in \mathcal{C}} M_{|C|}(\mathbb{C}); \qquad (x_{ij}) \mapsto ((x_{ij})_{i,j \in C})_{C \in \mathcal{C}}$$

4. the pure states of $E(\mathcal{R})$ are given by vector states $|v\rangle\langle v|$ for which the support of $v \in \ell^2(X)$ is $\mathcal{R}$-connected.

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Example

The operator systems of $p \times p$ band matrices with band width N.
The dual operator system consists of partially defined band matrices.

Operator systems associated to tolerance relations

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$$\mathrm{prop}(E(\mathcal{R}_\epsilon)) = \lceil diam(X)/\epsilon \rceil$$

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The pure states of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is ϵ-connected.

Chapter Five.

K-theory

K-theory for operator systems

[arXiv:2409.02773]

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Definition

A hermitian form x *in a unital operator system* E *is a selfadjoint element* $x \in M_n(E)$ *which is* non-degenerate *in the sense that there exists* $g > 0$ *such that for all pure and maximal ucp maps* $\phi : E \to B(\mathcal{H})$ *we have*

$$|\phi^{(n)}(x)| \geq g \cdot \mathrm{id}_{\mathcal{H}}^{\oplus n}$$

In other words, x *should have a gap* g *in each boundary representation*

We will write $H(E, n)$ for all hermitian forms in $M_n(E)$.

Proposition

An element $x \in M_n(E)$ is non-degenerate if and only if $\imath_E^{(n)}(x)$ is an invertible element in the C^*-envelope $C^*_{env}(E)$.

This is a consequence of the realization of the C^*-envelope in [Davidson–Kennedy]

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Examples:

1. Hermitian forms (à la Witt) on a fgp right module pA^n over a C^*-algebra A: described by invertible elements $x = h + (1 - p) \in M_n(A)$ with $h \in pM_n(A)p$.

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2. Projections p in operator systems à la Araiza–Russell are precisely projections in the C^*-envelope: $x = e - 2p$ is a hermitian form.

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4. Spectral compressions of projections in C^*-algebra: $x = PYP$ with $Y = 1 - 2p$ provided $\|[P, p]\|$ sufficiently small.

The invariants and K-theory

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and with the map $\imath_{nm}([x] = x \oplus e_{m-n}$ we have

$$\begin{array}{ccc} \mathcal{V}(\mathbb{C}, n) & \xrightarrow{\imath_{nm}} & \mathcal{V}(\mathbb{C}, m) \\ & {\scriptstyle \rho_n} \searrow \quad \swarrow {\scriptstyle \rho_m} & \\ & \mathbb{N} & \end{array}$$

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In general, we consider

$$\mathcal{V}(E) = \varinjlim \mathcal{V}(E,n)$$

and $K_0(E)$ is the corresponding Grothendieck group (with identity $[e]$ and addition $'\oplus'$)

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- Stability: we define a map $\imath_n : M_n(E) \to M_n(M_2(E))$ by

$$\imath_n(x) = \left(\begin{array}{cc|cc|c|cc} x_{11} & 0 & x_{12} & 0 & \cdots & x_{1n} & 0 \\ 0 & e & 0 & 0 & & 0 & 0 \\ \hline x_{21} & 0 & x_{22} & 0 & \cdots & \multicolumn{2}{c}{\vdots} \\ 0 & 0 & 0 & e & & & \\ \hline \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c}{\vdots} \\ \hline x_{n1} & 0 & \multicolumn{2}{c|}{\cdots} & \cdots & x_{nn} & 0 \\ 0 & 0 & & & & 0 & e \end{array}\right)$$

so that $\imath_n(x) \sim x$ (Whitehead). This allows to show $K_0(E) \cong K_0(M_2(E))$.

Non-unital operator systems and stability

The unitization [Werner, 2002] of a non-unital operator system E is given by the $*$-vector space $E^+ = E \oplus \mathbb{C}$ with matrix order structure:

$$(x, A) \geq 0 \text{ iff } A \geq 0 \text{ and } \phi(A_\epsilon^{-1/2} x A_\epsilon^{-1/2}) \geq -1$$

for all $\epsilon > 0$ and noncommutative states $\phi \in \mathcal{S}_n(E)$, and where $A_\epsilon = \epsilon \mathbb{I}_n + A$.

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3. The map $\kappa_{1\infty} : K_0(E) \to K_0(\mathcal{K} \otimes E)$ is an isomorphism:
   - injective: homotopy in $H((\mathcal{K} \otimes E)^+, n)$ compressed to homotopy in $H((M_N(E)^+, n)$.
   - surjective: approximation by finite-rank operators in norm is still hermitian form.

□

Chapter Six.

Applications

Example: Toeplitz matrices

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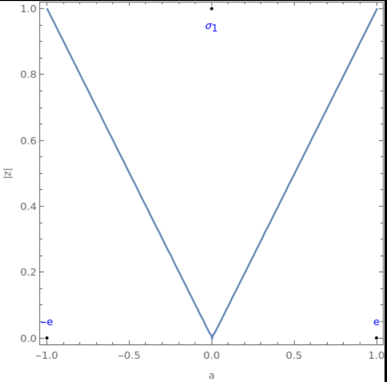

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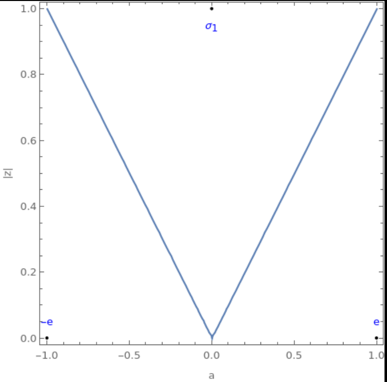


- $\mathcal{V}(C(S^1)^{(2)}, 1) \cong \{[-e], [\sigma_1], [e]\}$
- However, $\sigma_1 \oplus \sigma_1 \sim e \oplus (-e)$ in $H(C(S^1)^{(2)}, 2)$:

$$h(t) = \begin{pmatrix} (1-t)\sigma_1 + te & it(t-1)\sigma_2 \\ -it(t-1)\sigma_2 & (1-t)\sigma_1 - te \end{pmatrix}$$

with $\det h(t) > 0$.

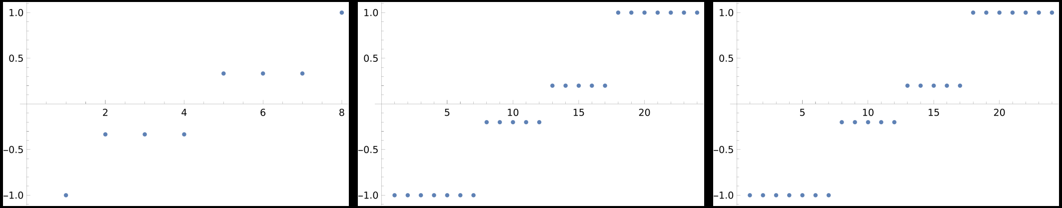
Example: spectral localizer on the 2-sphere

- Consider the Bott projection:

$$p = \frac{1}{2}\begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix} \in M_2(C(S^2))$$

  for $x^2 + y^2 + z^2 = 1$.
- Dirac operator $D_{S^2} \rightsquigarrow$ spinorial harmonics $Y^{\pm}_{lm}$ [GVF] with eigenvalues $\pm(l + 1/2)$ $(l = 1/2, 3/2, \ldots)$.
- Spectral projections P_ρ onto $\mathrm{span}_{\mathbb{C}}\{Y^{\pm}_{lm}\}_{l\leq\rho} \subset L^2(\mathcal{S}_{S^2})$.
- We obtain a compression $P_\rho Y P_\rho$ of the hermitian form $Y = 1 - 2p$ on $\mathbb{T}^2$ corresponding to p.

Eigenvalues for $P_\rho Y P_\rho$ (Bott) for $\rho = 1/2, 3/2, 5/2$



Spectral localizer

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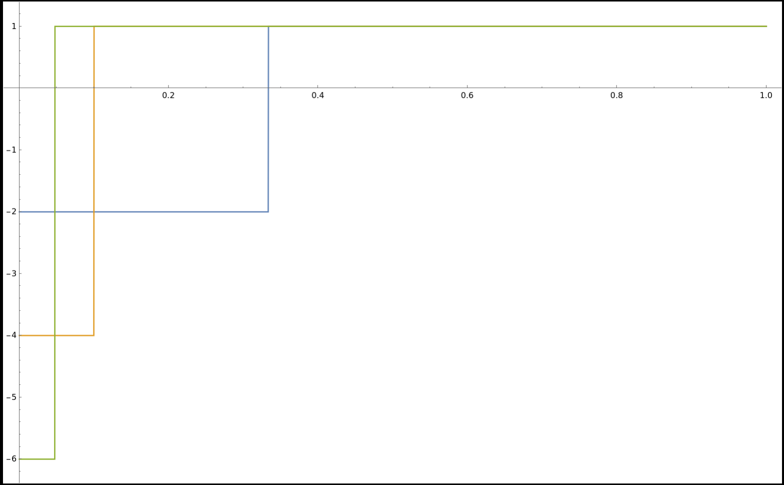
In general they show that for suitable κ and ρ the index pairing can be computed as the signature of this matrix:

$$\boxed{\text{Index } pD^{+}p = \frac{1}{2}\text{Sig } L_{\kappa,\rho}}$$

Signature of spectral localizer $L_{\kappa,\rho}$ for Bott (as a function of κ for $\rho = 1/2, 3/2, 5/2$)



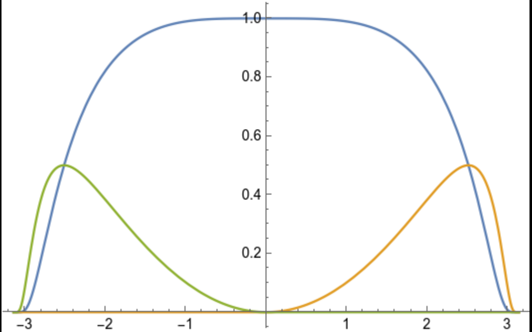
Example: spectral localizer on the 2-torus

Loring considered (in his thesis!) analogues of the Powers–Rieffel projections:

$$p = \begin{pmatrix} f & g + hU \\ g + hU^* & 1 - f \end{pmatrix} \in M_2(C^\infty(\mathbb{T}^2))$$

with U a unitary in the second variable, and f, g, h real-valued smooth functions in the first variable, satisfying

$$gh = 0, \qquad g^2 + h^2 = f - f^2.$$



Spectral truncations on $\mathbb{T}^2$

- We now consider spectral truncations $P = P_\rho$ onto $\ell^2\{\vec{n} \in \mathbb{Z}^2 : \|\vec{n}\| \leq \rho\} \subseteq l^2(\mathbb{Z}^2)$.

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- We obtain a compression PYP of the hermitian form $Y = 1 - 2p$ on $\mathbb{T}^2$ corresponding to p:

$$PYP = \begin{pmatrix} P - 2PfP & -2PgP - 2PhUP \\ -2PgP - 2PhU^*P & -P + 2PfP \end{pmatrix} \in M_2(PC^\infty(\mathbb{T}^2)P)$$

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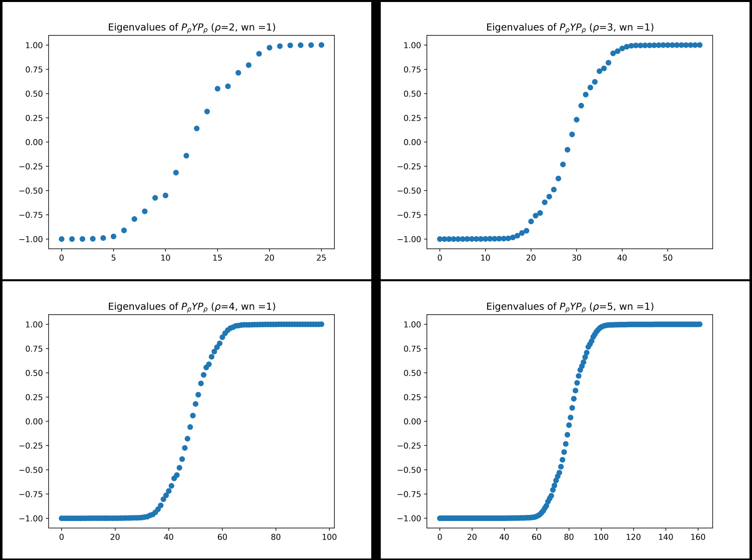
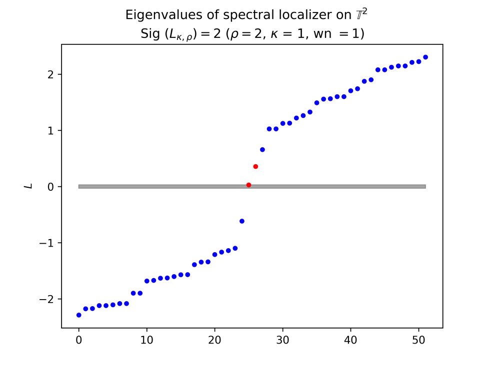
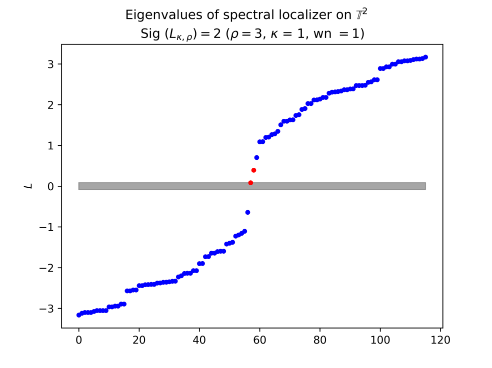
Simulations: eigenvalues of PYP for $U(t_2) = e^{it_2}$

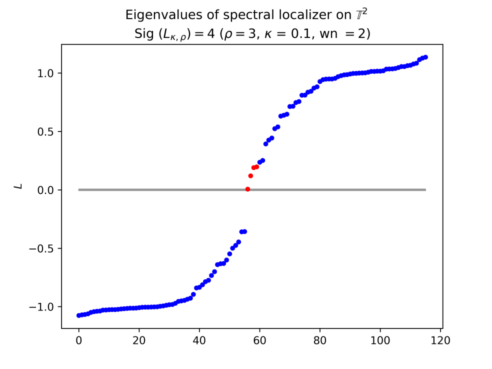


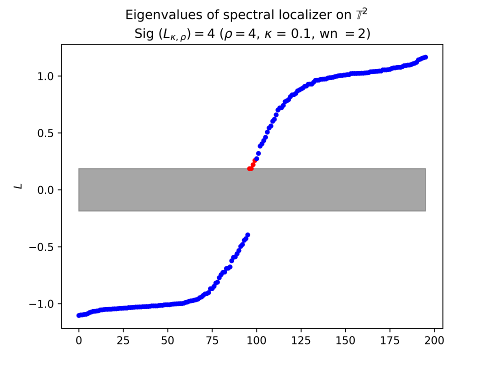
Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{it_2}$





Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{2it_2}$





Summary and outlook

- Noncommutative geometry: metric aspect
- Operator systems: from (spectral) truncations
- Duality of operator systems: state spaces
- New invariants: propagation number, K-theory
  - Higher K-group invariants [arXiv:2411.02981]:

$$\mathcal{V}_1^\delta(E,n) = \left\{ x \in M_n(E) : \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \text{ has spectral gap } \delta \right\} /_{\sim_n}$$

    and, more generally, $\mathcal{V}_p^\delta(E,n) := H^\delta(E \otimes \mathbb{C}l_p^{(1)}, n)/_{\sim_n}$.
  - Formal periodicity: $K_{2m}(E) = K_0(E)$ and $K_{2m+1}(E) = K_1(E)$.

- Persistence for projective systems:

$$A \to \cdots \to E_k \to E_{k+1} \to \cdots$$

  - When does $[H] \in K_0(A)$ induce a (non-trivial) class in $K_0(E_k)$ for some k?
  - Can use spectral localizer (*cf.* 2-torus) to check non-triviality.
- Persistence for inductive systems:

$$\cdots \to E_{k-1} \to E_k \to \cdots \to A$$

  - When do we have that $[x] \in K_0(E_k)$ persists to induce a (non-trivial) class in $K_0(A)$?
  - Can we extract invariants of A from the invariants $\{\mathcal{V}(E_k, k)\}_k$?
  - Relation to quantitative K-theory [Oyono-Oyono–Yu] and NF/CPC*-systems [Blackadar–Kirchberg, Courtney–Winter]