Noncommutative geometry and operator systems

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Chapter One.

Motivation

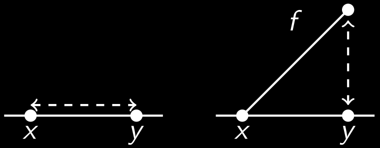
Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

- Distance $d(x, y)$ between two points is usually defined as
  the **smallest** *of the arclengths (computed using the metric) of curves connecting x and y.*

- But it can also be defined as
  the **largest** *of differences $|f(x) - f(y)|$ for functions f with gradient $|\nabla f| \leq 1$.*

$$\boxed{d(x, y) = \sup_{\|[D_M, f]\| \leq 1} |\delta_x(f) - \delta_y(f)|}$$



Combination $(C^\infty(M), L^2(S_M), D_M)$
allows for reconstruction of geometry

Spectral triples

More generally, we consider a triple $(\mathcal{A}, \mathcal{H}, D)$

- a unital $*$-algebra $\mathcal{A}$
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A}$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ of norm 1
- Distance function on state space $S(\mathcal{A})$ of $\mathcal{A}$:

$$d_D(\phi, \psi) = \sup_{a \in \mathcal{A}} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

These notions continue to make sense when we replace $\mathcal{A}$ by any self-adjoint vector space $\mathcal{E}$ of bounded operators on $\mathcal{H}$ that contains the unit, a so-called *operator system*.

Spectral data: $(\mathcal{A}, \mathcal{H}, D)$

- The mathematical reformulation of geometry in terms of spectral data (global analysis) requires the knowledge of the full Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.
- Also computer simulations are inevitably finite-dimensional, but still index computations can be done (spectral localizer).

Mathematically, this is realized by **spectral truncations** of $(A, \mathcal{H}, D)$:

- $\mathcal{H} \mapsto P\mathcal{H}$, spectral projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is <u>not an algebra</u> any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$

Chapter Two.

Operator Systems

Abstract operator systems

Definition

We say that a $*$-vector space is matrix ordered *if*

1. *for each n we are given a cone of positive elements $M_n(E)_+$ in $M_n(E)_h$,*
2. $M_n(E)_+ \cap (-M_n(E)_+) = \{0\}$ *for all n,*
3. *for every m, n and $A \in M_{mn}(\mathbb{C})$ we have that $AM_n(E)_+A^* \subseteq M_m(E)_+$.*

We call $e \in E_h$ an *order unit* for E if for each $x \in E_h$ there is a $t > 0$ such that $-te \leq x \leq te$. It is called an *Archimedean order unit* if $-te \leq x$ for all $t > 0$ implies that $x \geq 0$.

Definition

An (abstract) operator system *is given by a matrix-ordered $*$-vector space E with an order unit e such that for all n $e^{\oplus n}$ is an Archimedean order unit for $M_n(E)$.*

Maps between operator systems E, F are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

Isomorphisms are complete order isomorphisms

C^*-envelope of a unital operator system

[Arveson, 1969]

Hamana: existence and uniqueness in 1979; realized á la Arveson as direct sum of all boundary representations [Dritschel–McCullough 2005, Arveson 2008, Davidson–Kennedy 2015]

A C^*-extension $\kappa : E \to A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of a unital operator system is a C^*-extension $\kappa : E \to A$ with the following universal property:

$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & \searrow^{\lambda} & \uparrow{\scriptstyle \exists! \rho} \\ & & B \end{array}$$

Example: operator system $C_{\mathrm{harm}}(\overline{\mathbb{D}})$ of continuous harmonic functions with C^*-envelope $C(S^1)$.

Chapter Three.

S^1

Example: spectral truncation of the circle [Connes-vS, 2020]

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is the operator system of Toeplitz matrices:

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

- States are defined as unital positive linear functionals.

We have: $C^*_{\mathrm{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.
- The C^*-envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z}) \cong C(S^1)$

Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. *The extreme rays in* $C(S^1)_+^{(n)}$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n / S_n$.

Curiosities on Toeplitz matrices

Theorem (Carathéodory)

Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ iff $T = V \Delta V^*$ with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \qquad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \dots, d_n \geq 0$ and $\lambda_1, \dots, \lambda_n \in S^1$.

General results on GH-convergence

Definition

Let $\{(\mathcal{E}_n, \mathcal{H}_n, D_n)\}_n$ be a sequence of operator system spectral triples and let $(\mathcal{E}, \mathcal{H}, D)$ be an operator system spectral triple. An C^1-approximate order isomorphism for this set of data is given by linear maps $R_n : E \to E_n$ and $S_n : E_n \to E$ for any n such that the following three condition hold:

1. *R_n, S_n are positive, unital, contractive and Lipschitz-contractive*
2. *there exist sequences γ_n, γ_n' both converging to zero such that*

$$\|S_n \circ R_n(a) - a\| \leq \gamma_n \|[D, a]\|,$$
$$\|R_n \circ S_n(h) - h\| \leq \gamma_n' \|[D_n, h]\|.$$

Theorem

If (R_n, S_n) is a C^1-approximate order isomorphism for $(\mathcal{E}_n, \mathcal{H}_n, D_n)$ and $(\mathcal{E}, \mathcal{H}, D)$, then the state spaces $(\mathcal{S}(E_n), d_{E_n})$ converge to $(\mathcal{S}(E), d_E)$ in Gromov–Hausdorff distance.

Spectral truncations and convergence to the circle

- The map $R_n : C(S^1) \to C(S^1)^{(n)}$ given by compression with P_n allows to pull-back states from $C(S^1)^{(n)}$ to the circle
- There is a C^1-approximate order inverse $S_n : C(S^1)^{(n)} \to C(S^1)$:

$$R_n(S_n(T)) = T_n \odot T; \qquad S_n(R_n(f)) = F_n * f$$

  in terms of a Schur product with a matrix T_n and the convolution with the Fejér kernel F_n:

Proposition (vS21, Hekkelman 2021)

The sequence of state spaces $\{(\mathcal{S}(C(S^1)^{(n)}), d_n)\}$ converges to $(\mathcal{S}(C(S^1)), d_{S^1})$ in Gromov–Hausdorff distance.

Other examples: cubic truncations of $\mathbb{T}^d$ [Berendschot 2019], fuzzy spheres [Rieffel 2000], quantum spheres [Aguilar–Kaad–Kyed 2021], Fourier truncations [Rieffel 2022], spectral truncations of $\mathbb{T}^d$ [Leimbach 2023], Peter–Weyl truncations [Gaudillot–Estrada 2024, Leimbach 2024],...

Chapter Four.

Bonds

Operator systems, groupoids and bonds (aka a positivity domain)

Definition (Connes-vS, 2021)

A bond is a triple (G, ν, Ω) consisting of a locally compact groupoid G, a Haar system $\nu = \{\nu_x\}$ and an open symmetric subset $\Omega \subseteq G$ containing the units $G^{(0)}$.

Proposition

Let (Ω, G, ν) be a bond. The closure of the subspace $C_c(\Omega) \subseteq C_c(G)$ in the C^*-algebra $C^*(G)$ is a (possibly non-unital) operator system.

Example

1. *Consider Ω in a l.c. Lie group $G \rightsquigarrow$ Fourier truncations (á la Rieffel) in $C^*(G)$*
2. *Consider $\Omega_n = (-n, n) \subset \mathbb{Z} \rightsquigarrow$ Fejér–Riesz system $C^*(\mathbb{Z})_{(n)} \cong (C(S^1)^{(n)})^d$.*
3. *Consider $\Omega_n = (-n, n) \subseteq C_m$ (so modulo m). The operator system consists of banded $m \times m$ circulant matrices of band width n.*

Tolerance relations on finite sets [Gielen–vS, 2022]

Let X be a finite set and $\mathcal{R} \subseteq X \times X$ a symmetric reflexive relation on X

1. The C^*-envelope of $E(\mathcal{R})$ is $\mathcal{K}(\ell^2(X)) \cong M_{|X|}(\mathbb{C})$ and $\mathrm{prop}(E(\mathcal{R})) = \mathrm{diam}(\mathcal{R})$.
2. If $\mathcal{R}$ is a chordal graph, then $E(\mathcal{R})^d \cong E(\mathcal{R})$ as a vector space, but with order structure given by being partially positive.
3. Concrete realization of $E(\mathcal{R})^d$ in terms of cliques C in $\mathcal{R}$:

$$\Phi : E(\mathcal{R})^d \to \bigoplus_{C \in \mathcal{C}} M_{|C|}(\mathbb{C}); \qquad (x_{ij}) \mapsto ((x_{ij})_{i,j \in C})_{C \in \mathcal{C}}$$

4. the pure states of $E(\mathcal{R})$ are given by vector states $|v\rangle\langle v|$ for which the support of $v \in \ell^2(X)$ is $\mathcal{R}$-connected.

Example

The operator systems of $p \times p$ band matrices with band width N.
The dual operator system consists of partially defined band matrices.

Operator systems associated to tolerance relations

- Key motivating example: a metric space (X, d) with the relation

$$\boxed{\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}}$$

- If (X, μ) is a measure space and $\mathcal{R}_\epsilon \subseteq X \times X$ an open subset we obtain the operator system $E(\mathcal{R}_\epsilon)$ as the closure of integral operators with support in $\mathcal{R}_\epsilon$. Note that $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$

Proposition

Let X be a complete, locally compact path metric measure space with a measure of full support. Then $C^*_{env}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$ and

$$\mathrm{prop}(E(\mathcal{R}_\epsilon)) = \lceil diam(X)/\epsilon \rceil$$

The pure states of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is ϵ-connected.

Chapter Five.

K-theory

K-theory for operator systems

[arXiv:2409.02773]

A key invariant of C^*-algebras is K-theory. Is there an analogue for operator systems?

- Need notion of projection (*cf.* Araiza–Russell) or invertible selfadjoint elements
- It should capture the spectral localizer of Loring, Schulz-Baldes, and others
- It should be invariant under Morita equivalence [EKT]

Definition

A hermitian form x *in a unital operator system* E *is a selfadjoint element* $x \in M_n(E)$ *which is* non-degenerate *in the sense that there exists* $g > 0$ *such that for all pure and maximal ucp maps* $\phi : E \to B(\mathcal{H})$ *we have*

$$|\phi^{(n)}(x)| \geq g \cdot \mathrm{id}_{\mathcal{H}}^{\oplus n}$$

In other words, x *should have a gap* g *in each boundary representation*

We will write $H(E, n)$ for all hermitian forms in $M_n(E)$.

Proposition

An element $x \in M_n(E)$ is non-degenerate if and only if $\imath_E^{(n)}(x)$ is an invertible element in the C^*-envelope $C^*_{env}(E)$.

This is a consequence of the realization of the C^*-envelope in [Davidson–Kennedy]

Examples:

1. Hermitian forms (à la Witt) on a fgp right module pA^n over a C^*-algebra A: described by invertible elements $x = h + (1 - p) \in M_n(A)$ with $h \in pM_n(A)p$.
2. Projections p in operator systems à la Araiza–Russell are precisely projections in the C^*-envelope: $x = e - 2p$ is a hermitian form.
3. Similarly, ϵ-projections in quantitative K-theory [Oyono-Oyono–Yu] define hermitian forms.
4. Spectral compressions of projections in C^*-algebra: $x = PYP$ with $Y = 1 - 2p$ provided $\|[P, p]\|$ sufficiently small.

The invariants and K-theory

$$\boxed{\mathcal{V}(E, n) = H(E, n)/_{\sim_n}}$$

Example:

$$\mathcal{V}(\mathbb{C}, n) \cong \{-n, -n+2, \ldots, n\}$$

and with the map $\imath_{nm}([x] = x \oplus e_{m-n}$ we have

$$\begin{array}{ccc} \mathcal{V}(\mathbb{C}, n) & \xrightarrow{\imath_{nm}} & \mathcal{V}(\mathbb{C}, m) \\ & {\scriptstyle\rho_n} \searrow \quad \swarrow {\scriptstyle\rho_m} & \\ & \mathbb{N} & \end{array}$$

In general, we consider

$$\mathcal{V}(E) = \varinjlim \mathcal{V}(E, n)$$

and $K_0(E)$ is the corresponding Grothendieck group (with identity $[e]$ and addition $'\oplus'$)

Properties of K_0

- For C^*-algebras we obtain usual K-theory via the map $[x] \mapsto [p = \frac{1}{2}(1 - x|x|^{-1})]$.
- Stability: we define a map $\imath_n : M_n(E) \to M_n(M_2(E))$ by

$$\imath_n(x) = \left(\begin{array}{cc|cc|c|cc} x_{11} & 0 & x_{12} & 0 & \cdots & x_{1n} & 0 \\ 0 & e & 0 & 0 & & 0 & 0 \\ \hline x_{21} & 0 & x_{22} & 0 & \cdots & \multicolumn{2}{c}{\vdots} \\ 0 & 0 & 0 & e & & & \\ \hline \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c}{\vdots} \\ \hline x_{n1} & 0 & \multicolumn{2}{c|}{\cdots} & \cdots & x_{nn} & 0 \\ 0 & 0 & & & & 0 & e \end{array}\right)$$

so that $\imath_n(x) \sim x$ (Whitehead). This allows to show $K_0(E) \cong K_0(M_2(E))$.

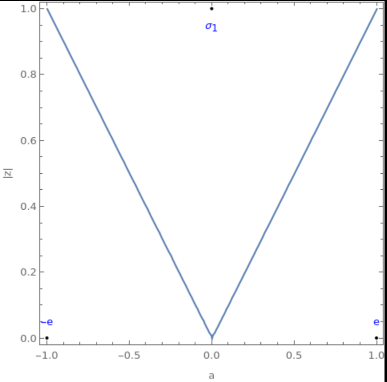
Chapter Six.

Applications

Example: Toeplitz matrices

- Consider the operator system $C(S^1)^{(2)}$ of 2×2 Toeplitz matrices.
- Hermitian forms in $H(C(S^1)^{(2)}, 1)$ are matrices of the form

$$T = \begin{pmatrix} a & z \\ \overline{z} & a \end{pmatrix}; \qquad a^2 - |z| \neq 0.$$



- $\mathcal{V}(C(S^1)^{(2)}, 1) \cong \{[-e], [\sigma_1], [e]\}$
- However, $\sigma_1 \oplus \sigma_1 \sim e \oplus (-e)$ in $H(C(S^1)^{(2)}, 2)$:

$$h(t) = \begin{pmatrix} (1-t)\sigma_1 + te & it(t-1)\sigma_2 \\ -it(t-1)\sigma_2 & (1-t)\sigma_1 - te \end{pmatrix}$$

with $\det h(t) > 0$.

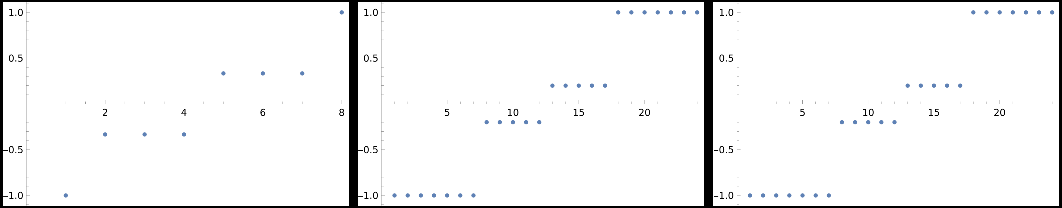
Example: spectral localizer on the 2-sphere

- Consider the Bott projection:

$$p = \frac{1}{2}\begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix} \in M_2(C(S^2))$$

  for $x^2 + y^2 + z^2 = 1$.
- Dirac operator $D_{S^2} \rightsquigarrow$ spinorial harmonics $Y^{\pm}_{lm}$ [GVF] with eigenvalues $\pm(l + 1/2)$ $(l = 1/2, 3/2, \ldots)$.
- Spectral projections P_ρ onto $\mathrm{span}_{\mathbb{C}}\{Y^{\pm}_{lm}\}_{l\leq\rho} \subset L^2(\mathcal{S}_{S^2})$.
- We obtain a compression $P_\rho Y P_\rho$ of the hermitian form $Y = 1 - 2p$ on $\mathbb{T}^2$ corresponding to p.

Eigenvalues for $P_\rho Y P_\rho$ (Bott) for $\rho = 1/2, 3/2, 5/2$



Spectral localizer

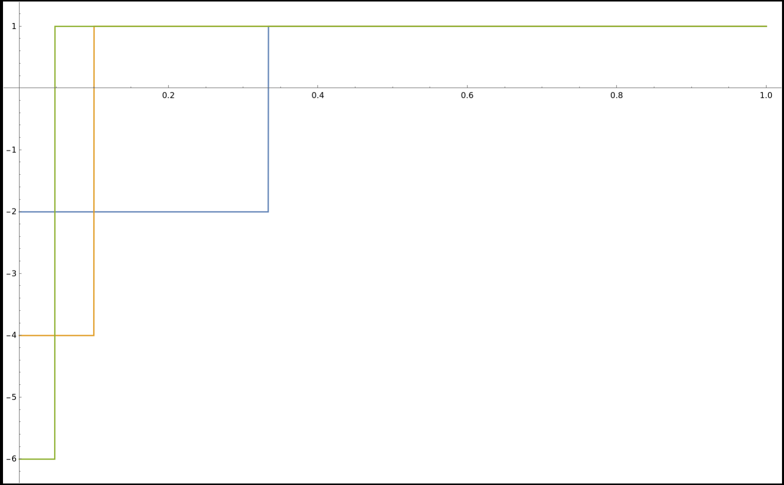
- The *spectral localizer* of Loring and Schulz-Baldes is given by the following matrix:

$$L_{\kappa,\rho} = \begin{pmatrix} -PYP & \kappa PD^+P \\ \kappa PD^-P & PYP \end{pmatrix}$$

In general they show that for suitable κ and ρ the index pairing can be computed as the signature of this matrix:

$$\boxed{\text{Index } pD^+p = \frac{1}{2}\text{Sig } L_{\kappa,\rho}}$$

Signature of spectral localizer $L_{\kappa,\rho}$ for Bott (as a function of κ for $\rho = 1/2, 3/2, 5/2$)



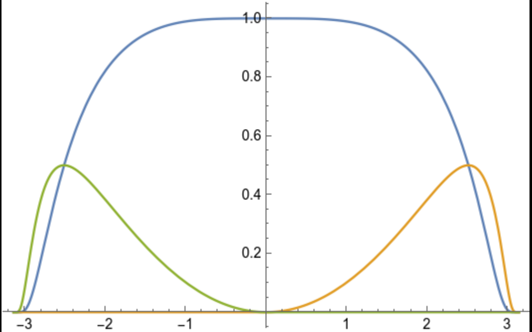
Example: spectral localizer on the 2-torus

Loring considered (in his thesis!) analogues of the Powers–Rieffel projections:

$$p = \begin{pmatrix} f & g + hU \\ g + hU^* & 1 - f \end{pmatrix} \in M_2(C^\infty(\mathbb{T}^2))$$

with U a unitary in the second variable, and f, g, h real-valued smooth functions in the first variable, satisfying

$$gh = 0, \qquad g^2 + h^2 = f - f^2.$$



Spectral truncations on $\mathbb{T}^2$

- We now consider spectral truncations $P = P_\rho$ onto $\ell^2\{\vec{n} \in \mathbb{Z}^2 : \|\vec{n}\| \leq \rho\} \subseteq l^2(\mathbb{Z}^2)$.
- We obtain a compression PYP of the hermitian form $Y = 1 - 2p$ on $\mathbb{T}^2$ corresponding to p:

$$PYP = \begin{pmatrix} P - 2PfP & -2PgP - 2PhUP \\ -2PgP - 2PhU^*P & -P + 2PfP \end{pmatrix} \in M_2(PC^\infty(\mathbb{T}^2)P)$$

  For suitable P these are hermitian forms $\rightsquigarrow [PYP] \in K_0(PC(\mathbb{T}^2)P)$.
- The *spectral localizer* of Loring and Schulz-Baldes is given by the following matrix:

$$L_{\kappa,\rho} = \begin{pmatrix} -PYP & \kappa PD^+P \\ \kappa PD^-P & PYP \end{pmatrix}$$

  In general they show that for suitable κ and ρ the index pairing can be computed as the signature of this matrix:

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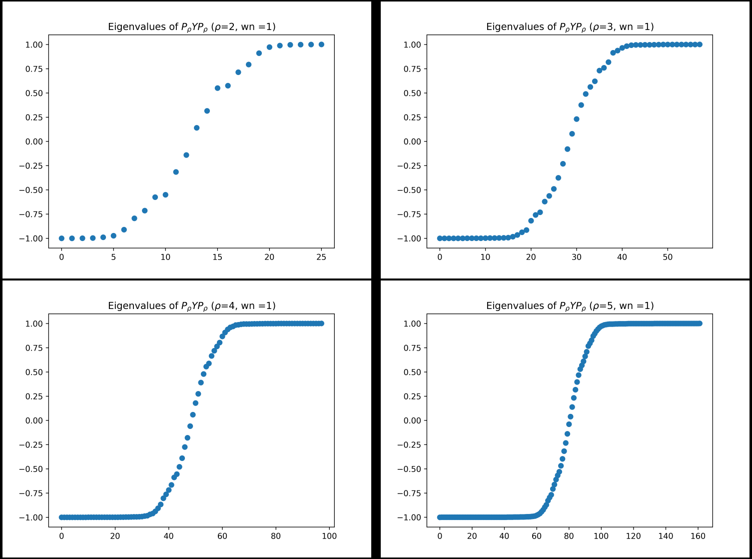
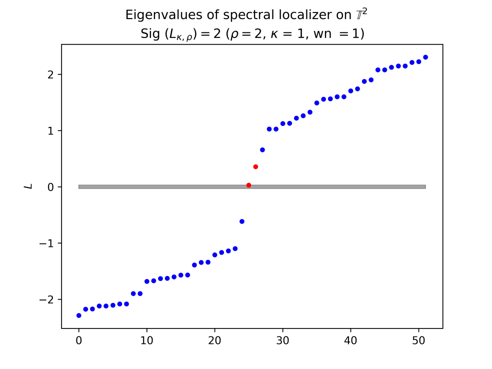
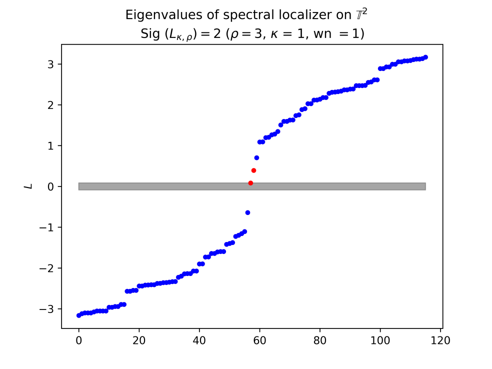
Simulations: eigenvalues of PYP for $U(t_2) = e^{it_2}$

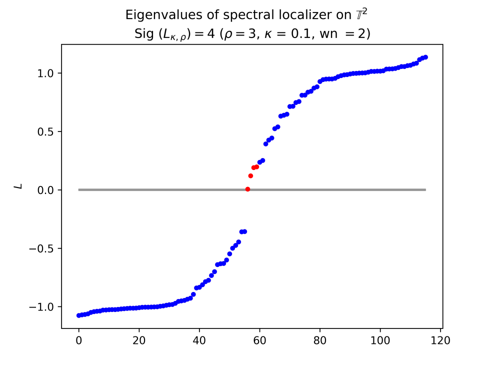


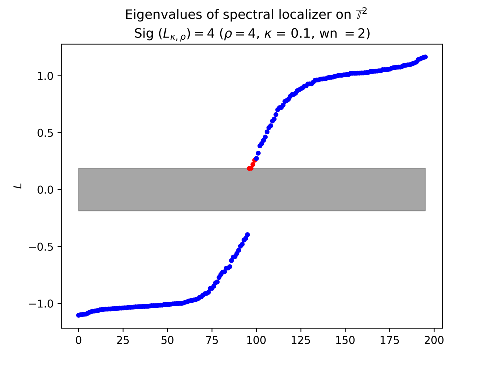
Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{it_2}$





Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{2it_2}$





Summary and outlook

- Noncommutative geometry: metric aspect
- Operator systems: from (spectral) truncations
- Duality of operator systems: state spaces
- New invariants: propagation number, K-theory
  - K-group invariants [arXiv:2409.02773]:

$$\mathcal{V}_0(E,n) = \{x = x^* \in M_n(E) : x \text{ is invertible } \} /_{\sim_n}$$

  - Grothendieck group $K_0(E)$ of $\varinjlim \mathcal{V}_0(E,n)$ is invariant under Morita equivalence.
  - Higher K-group invariants [arXiv:2411.02981, TAMS]:

$$\mathcal{V}_1^\delta(E,n) = \left\{ x \in M_n(E) : \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \text{ has spectral gap } \delta \right\} /_{\sim_n}$$

    and, more generally, $\mathcal{V}_p^\delta(E,n) := H^\delta(E \otimes \mathbb{C}l_p^{(1)}, n)/_{\sim_n}$.
  - Formal periodicity: $K_{2m}(E) = K_0(E)$ and $K_{2m+1}(E) = K_1(E)$.

- Persistence for projective systems:

$$A \to \cdots \to E_k \to E_{k+1} \to \cdots$$

  - When does $[H] \in K_0(A)$ induce a (non-trivial) class in $K_0(E_k)$ for some k?
  - Can use spectral localizer (*cf.* 2-torus) to check non-triviality.
- Persistence for inductive systems:

$$\cdots \to E_{k-1} \to E_k \to \cdots \to A$$

  - When do we have that $[x] \in K_0(E_k)$ persists to induce a (non-trivial) class in $K_0(A)$?
  - Can we extract invariants of A from the invariants $\{\mathcal{V}(E_k, k)\}_k$?
  - Relation to quantitative K-theory [Oyono-Oyono–Yu] and NF/CPC*-systems [Blackadar–Kirchberg, Courtney–Winter]