A generalization of K-theory to operator systems

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Chapter One.

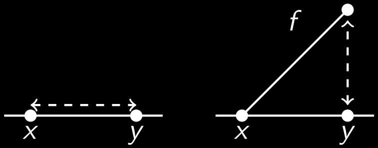
Motivation

Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

- Distance $d(x, y)$ between two points is usually defined as
  the **smallest** *of the arclengths (computed using the metric) of curves connecting x and y.*
- But it can also be defined as
  the **largest** *of differences $|f(x) - f(y)|$ for functions f with gradient $|\nabla f| \leq 1$.*

$$\boxed{d(x, y) = \sup_{\|[D_M, f]\| \leq 1} |\delta_x(f) - \delta_y(f)|}$$



Combination $(C^\infty(M), L^2(S_M), D_M)$
allows for reconstruction of geometry

Spectral triples

More generally, we consider a triple $(\mathcal{A}, \mathcal{H}, D)$

- a unital $*$-algebra $\mathcal{A}$
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A}$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ of norm 1
- Distance function on state space $S(\mathcal{A})$ of $\mathcal{A}$:

$$d_D(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$

These notions continue to make sense when we replace $\mathcal{A}$ by any self-adjoint vector space $\mathcal{E}$ of bounded operators on $\mathcal{H}$ that contains the unit, a so-called *operator system*.

Spectral data: $(\mathcal{A}, \mathcal{H}, D)$

- The mathematical reformulation of geometry in terms of spectral data (global analysis) requires the knowledge of the full Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.
- Also computer simulations are inevitably finite-dimensional, but still index computations can be done (spectral localizer).

Mathematically, this is realized by **spectral truncations** of $(A, \mathcal{H}, D)$:

- $\mathcal{H} \mapsto P\mathcal{H}$, spectral projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$

Chapter Two.

Operator
Systems

Abstract operator systems

Definition

We say that a $*$-vector space is matrix ordered *if*

1. *for each n we are given a cone of positive elements $M_n(E)_+$ in $M_n(E)_h$,*
2. $M_n(E)_+ \cap (-M_n(E)_+) = \{0\}$ *for all n,*
3. *for every m, n and $A \in M_{mn}(\mathbb{C})$ we have that $AM_n(E)_+A^* \subseteq M_m(E)_+$.*

We call $e \in E_h$ an *order unit* for E if for each $x \in E_h$ there is a $t > 0$ such that $-te \leq x \leq te$. It is called an *Archimedean order unit* if $-te \leq x$ for all $t > 0$ implies that $x \geq 0$.

Definition

An (abstract) operator system *is given by a matrix-ordered $*$-vector space E with an order unit e such that for all n $e^{\oplus n}$ is an Archimedean order unit for $M_n(E)$.*

Maps between operator systems E, F are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

Isomorphisms are complete order isomorphisms

C^*-envelope of a unital operator system

[Arveson, 1969]

Hamana: existence and uniqueness in 1979; realized á la Arveson as direct sum of all boundary representations [Dritschel–McCullough 2005, Arveson 2008, Davidson–Kennedy 2015]

A C^*-extension $\kappa : E \to A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of a unital operator system is a C^*-extension $\kappa : E \to A$ with the following universal property:

$$\begin{tikzcd} E \arrow[r, "\kappa"] \arrow[dr, "\lambda"'] & A \\ & B \arrow[u, dotted, two heads, "\exists! \rho"'] \end{tikzcd}$$

Example: operator system $C_{\mathsf{harm}}(\overline{\mathbb{D}})$ of continuous harmonic functions with C^*-envelope $C(S^1)$.

How far is an operator system from a C^*-algebra?

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E.

Definition

The propagation number $\text{prop}(E)$ *of* E *is defined as the smallest integer* n *such that* $(\imath_E(E))^{\circ n} \subseteq C^*_{env}(E)$ *is a* C^**-algebra.*

Returning to the harmonic functions in the disk we have $\text{prop}(C_{\text{harm}}(\overline{\mathbb{D}})) = 1$.

Proposition (Connes-vS, 2020; Pawlowska, 2024)

The propagation number is invariant under complete order isomorphisms, as well as under stable=Morita equivalence [EKT, 2019]:

$$prop(E) = prop(E \otimes_{min} \mathcal{K})$$

More generally [Koot, 2021], we have

$$prop(E \otimes_{\text{min}} F) = \max\{prop(E), prop(F)\}$$

Chapter Three.

S^1

Example: spectral truncation of the circle [Connes-vS, 2020]

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is the operator system of Toeplitz matrices:

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

- States are defined as unital positive linear functionals.

We have: $C^*_{\mathrm{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\mathrm{prop}(C(S^1)^{(n)}) = 2$ (for any n).

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.
- The C^*-envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z}) \cong C(S^1)$; propagation number ∞

Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. *The extreme rays in* $C(S^1)^{(n)}_+$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$.

Curiosities on Toeplitz matrices

Theorem (Carathéodory)

Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ iff $T = V \Delta V^*$ with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \qquad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \ldots, d_n \geq 0$ and $\lambda_1, \ldots, \lambda_n \in S^1$.

Chapter Five.

K-theory

K-theory for operator systems

[arXiv:2409.02773]

A key invariant of C^*-algebras is K-theory. Is there an analogue for operator systems?

- Need notion of projection (*cf.* Araiza–Russell) or invertible selfadjoint elements
- It should capture the spectral localizer of Loring, Schulz-Baldes, and others
- It should be invariant under Morita equivalence [EKT]

Definition

A hermitian form x *in a unital operator system* E *is a selfadjoint element* $x \in M_n(E)$ *which is* non-degenerate *in the sense that there exists* $g > 0$ *such that for all pure and maximal ucp maps* $\phi : E \to B(\mathcal{H})$ *we have*

$$|\phi^{(n)}(x)| \geq g \cdot \mathrm{id}_{\mathcal{H}}^{\oplus n}$$

In other words, x *should have a gap* g *in each boundary representation*

We will write $H(E,n)$ for all hermitian forms in $M_n(E)$.

Proposition

An element $x \in M_n(E)$ is non-degenerate if and only if $\imath_E^{(n)}(x)$ is an invertible element in the C^*-envelope $C^*_{env}(E)$.

This is a consequence of the realization of the C^*-envelope in [Davidson–Kennedy]

Examples:

1. Hermitian forms (à la Witt) on a fgp right module pA^n over a C^*-algebra A: described by invertible elements $x = h + (1-p) \in M_n(A)$ with $h \in pM_n(A)p$.
2. Projections p in operator systems à la Araiza–Russell are precisely projections in the C^*-envelope: $x = e - 2p$ is a hermitian form.
3. Similarly, ϵ-projections in quantitative K-theory [Oyono-Oyono–Yu] define hermitian forms.
4. Spectral compressions of projections in C^*-algebra: $x = PYP$ with $Y = 1 - 2p$ provided $\|[P,p]\|$ sufficiently small.

The invariants and K-theory

$$\boxed{\mathcal{V}(E,n) = H(E,n)/_{\sim_n}}$$

Example:

$$\mathcal{V}(\mathbb{C},n) \cong \{-n,-n+2,\ldots,n\}$$

and with the map $\imath_{nm}([x] = x \oplus e_{m-n}$ we have

$$\begin{tikzcd}
\mathcal{V}(\mathbb{C},n) \arrow[rr, "\imath_{nm}"] \arrow[dr, "\rho_n"'] & & \mathcal{V}(\mathbb{C},m) \arrow[dl, "\rho_m"] \\
& \mathbb{N} &
\end{tikzcd}$$

In general, we consider

$$\mathcal{V}(E) = \varinjlim \mathcal{V}(E,n)$$

and $K_0(E)$ is the corresponding Grothendieck group (with identity $[e]$ and addition $'\oplus'$)

Properties of K_0

- For C^*-algebras we obtain usual K-theory via the map $[x] \mapsto [p = \frac{1}{2}(1 - x|x|^{-1})]$.
- Stability: we define a map $\imath_n : M_n(E) \to M_n(M_2(E))$ by

$$\imath_n(x) = \left(\begin{array}{cc|cc|c|cc} x_{11} & 0 & x_{12} & 0 & \cdots & x_{1n} & 0 \\ 0 & e & 0 & 0 & & 0 & 0 \\ \hline x_{21} & 0 & x_{22} & 0 & \cdots & \multicolumn{2}{c}{\vdots} \\ 0 & 0 & 0 & e & & & \\ \hline \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c}{\vdots} \\ \hline x_{n1} & 0 & \multicolumn{2}{c|}{\cdots} & \cdots & x_{nn} & 0 \\ 0 & 0 & & & & 0 & e \end{array}\right)$$

so that $\imath_n(x) \sim x$ (Whitehead). This allows to show $K_0(E) \cong K_0(M_2(E))$.

Non-unital operator systems and stability

The unitization [Werner, 2002] of a non-unital operator system E is given by the $*$-vector space $E^+ = E \oplus \mathbb{C}$ with matrix order structure:

$$(x, A) \geq 0 \text{ iff } A \geq 0 \text{ and } \phi(A_\epsilon^{-1/2} x A_\epsilon^{-1/2}) \geq -1$$

for all $\epsilon > 0$ and noncommutative states $\phi \in \mathcal{S}_n(E)$, and where $A_\epsilon = \epsilon \mathbb{I}_n + A$.

$$\widetilde{\mathcal{V}}(E, n) := \left\{ (x, A) \in H(E^+, n) : A \sim_n \mathbb{I}_n \right\} / \sim_n$$

In the unital case, the isomorphism $E^+ \cong E \oplus \mathbb{C}$ given by $(x, A) \mapsto (x + Ae, A)$ yields that in this case

$$\widetilde{\mathcal{V}}(E, n) \cong \mathcal{V}(E, n).$$

Theorem

For a unital operator system E we have $K_0(\mathcal{K} \otimes E) \cong K_0(E)$.

Stability

Theorem

For a unital operator system E we have $K_0(\mathcal{K} \otimes E) \cong K_0(E)$.

Proof.

1. Realize stabilization by maps $\kappa_{NM} : M_N(E) \to M_M(E), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0_{M-N} \end{pmatrix}$
2. Commuting diagram:

$$\begin{tikzcd} \widetilde{V}(E,n) \arrow[r, "\kappa_{1N}"] \arrow[d, "\cong"'] & \widetilde{\mathcal{V}}(M_N(E),n) \arrow[d, "\cong"'] \\ \mathcal{V}(E,n) \arrow[r, "\imath_n", "\cong"'] & \mathcal{V}(M_N(E),n) \end{tikzcd}$$

3. The map $\kappa_{1\infty} : K_0(E) \to K_0(\mathcal{K} \otimes E)$ is an isomorphism:
   - injective: homotopy in $H((\mathcal{K} \otimes E)^+, n)$ compressed to homotopy in $H((M_N(E)^+, n)$.
   - surjective: approximation by finite-rank operators in norm is still hermitian form.

Chapter Six.

Applications

Example: $\mathcal{V}$ of the Toeplitz operator system $C(S^1)^{(n)}$

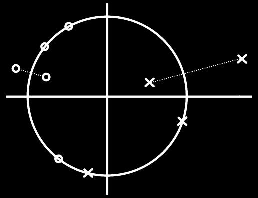
Lemma

The space $H(C(S^1)^{(n)}, 1)$ of non-degenerate hermitian Toeplitz matrices has at least $n+1$ path-connected components.

Proof.

1. Find a suitable generating function $f(z) = -\overline{f(\overline{z}^{-1})}$ for T, so that

$$f(z) = t_0^+ + t_1 z + t_2 z^2 + \cdots; \qquad f(z) = -t_0^- + t_{-1} z^{-1} + t_{-2} z^{-2} + \cdots$$

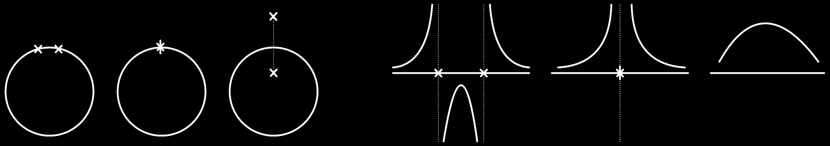


2. Show that the signature of T is equal to the Cauchy index of $f(z)$

Lemma

The set of non-degenerate hermitian Toeplitz matrices with fixed signature k is path-connected for any $k \in \{-n, -n+2, \dots, n\}$.

Proof.



$\square$

Proposition

The space $H(C(S^1)^{(n)}, 1)$ of non-degenerate hermitian Toeplitz matrices has $n+1$ path-connected components, hence $\mathcal{V}(C(S^1)^{(n)}, 1) \cong \{-n, -n+2, \dots, n\}$.

Example: spectral localizer on the 2-sphere

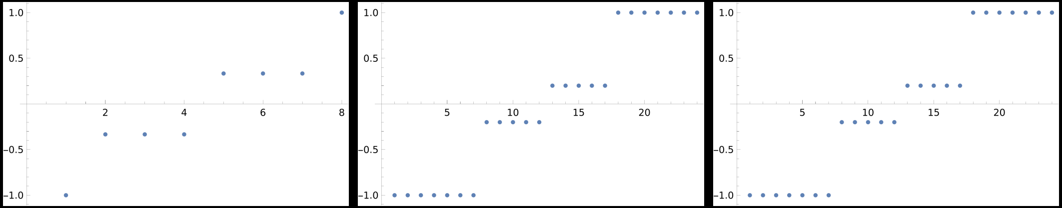
- Consider the Bott projection:

$$p = \frac{1}{2}\begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix} \in M_2(C(S^2))$$

for $x^2 + y^2 + z^2 = 1$.

- Dirac operator $D_{S^2} \rightsquigarrow$ spinorial harmonics $Y^{\pm}_{lm}$ [GVF] with eigenvalues $\pm(l + 1/2)$ $(l = 1/2, 3/2, \dots)$.
- Spectral projections P_ρ onto $\mathrm{span}_{\mathbb{C}}\{Y^{\pm}_{lm}\}_{l\leq\rho} \subset L^2(\mathcal{S}_{S^2})$.
- We obtain a compression $P_\rho Y P_\rho$ of the hermitian form $Y = 1 - 2p$ on $\mathbb{T}^2$ corresponding to p.

Eigenvalues for $P_\rho Y P_\rho$ (Bott) for $\rho = 1/2, 3/2, 5/2$



Spectral localizer

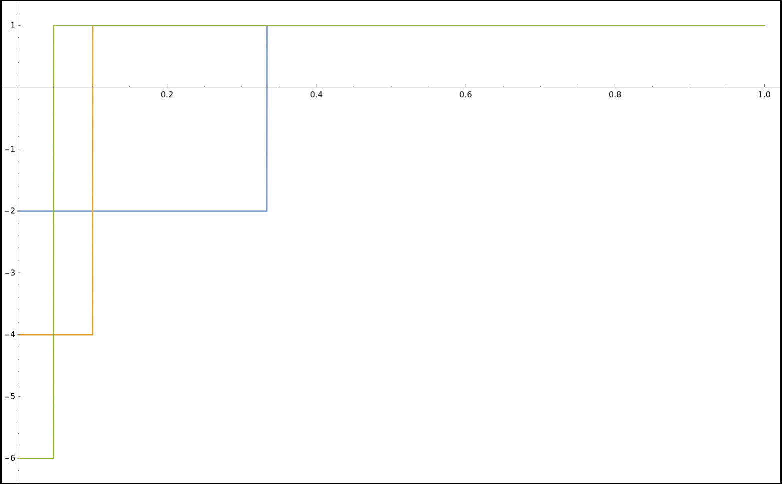
- The *spectral localizer* of Loring and Schulz-Baldes is given by the following matrix:

$$L_{\kappa,\rho} = \begin{pmatrix} -PYP & \kappa PD^+P \\ \kappa PD^-P & PYP \end{pmatrix}$$

  In general they show that for suitable κ and ρ the index pairing can be computed as the signature of this matrix:

$$\boxed{\text{Index } pD^+p = \frac{1}{2}\text{Sig } L_{\kappa,\rho}}$$

Signature of spectral localizer $L_{\kappa,\rho}$ for Bott (as a function of κ for $\rho = 1/2, 3/2, 5/2$)



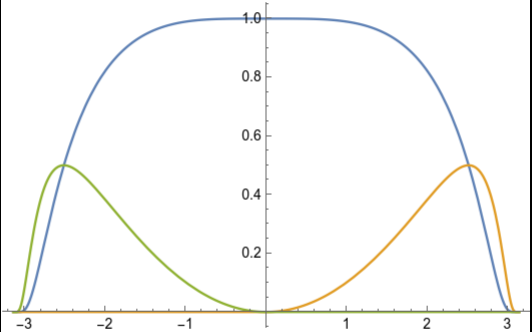
Example: spectral localizer on the 2-torus

Loring considered (in his thesis!) analogues of the Powers–Rieffel projections:

$$p = \begin{pmatrix} f & g + hU \\ g + hU^* & 1 - f \end{pmatrix} \in M_2(C^\infty(\mathbb{T}^2))$$

with U a unitary in the second variable, and f, g, h real-valued smooth functions in the first variable, satisfying

$$gh = 0, \qquad g^2 + h^2 = f - f^2.$$



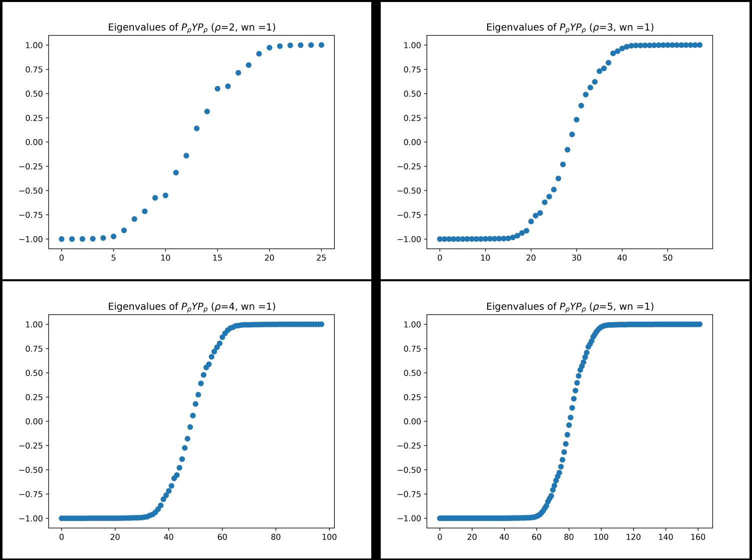
Spectral truncations on $\mathbb{T}^2$

- We now consider spectral truncations $P = P_\rho$ onto $\ell^2\{\vec{n} \in \mathbb{Z}^2 : \|\vec{n}\| \leq \rho\} \subseteq l^2(\mathbb{Z}^2)$.
- We obtain a compression PYP of the hermitian form $Y = 1 - 2p$ on $\mathbb{T}^2$ corresponding to p:

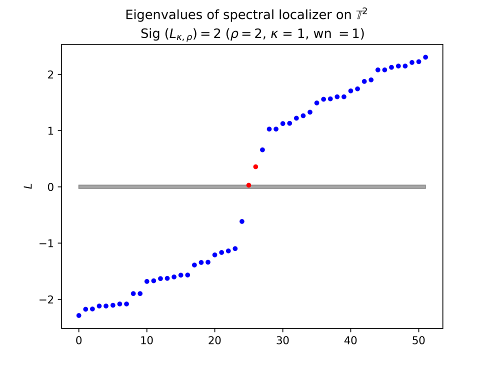
$$PYP = \begin{pmatrix} P - 2PfP & -2PgP - 2PhUP \\ -2PgP - 2PhU^*P & -P + 2PfP \end{pmatrix} \in M_2(PC^\infty(\mathbb{T}^2)P)$$

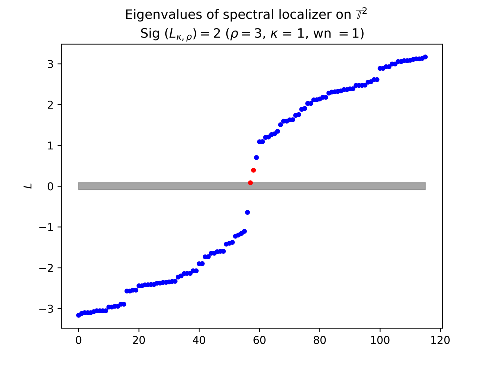
For suitable P these are hermitian forms $\rightsquigarrow [PYP] \in K_0(PC(\mathbb{T}^2)P)$.

Simulations: eigenvalues of PYP for $U(t_2) = e^{it_2}$

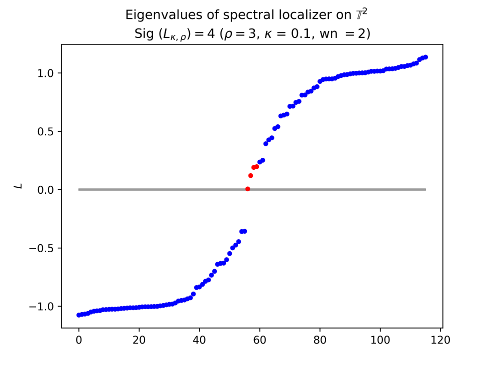


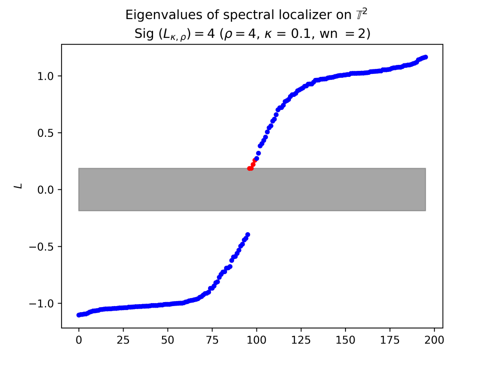
Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{it_2}$





Simulations: eigenvalues of $L_{\kappa,\rho}$ for $U(t_2) = e^{2it_2}$





Summary and outlook

- Noncommutative geometry: metric aspect
- Operator systems: from (spectral) truncations
- Duality of operator systems: state spaces
- New invariants: propagation number, K-theory
  - Higher K-group invariants [arXiv:2411.02981, Trans. AMS]:

$$\mathcal{V}_1^\delta(E,n) = \left\{ x \in M_n(E) : \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \text{ has spectral gap } \delta \right\} /_{\sim_n}$$

    and, more generally, $\mathcal{V}_p^\delta(E,n) := H^\delta(E \otimes \mathbb{C}l_p^{(1)}, n)/_{\sim_n}$.
  - Formal periodicity: $K_{2m}(E) = K_0(E)$ and $K_{2m+1}(E) = K_1(E)$.

- Persistence for projective systems:

$$A \to \cdots \to E_k \to E_{k+1} \to \cdots$$

  - When does $[H] \in K_0(A)$ induce a (non-trivial) class in $K_0(E_k)$ for some k?
  - Can use spectral localizer (*cf.* 2-torus) to check non-triviality.
- Persistence for inductive systems:

$$\cdots \to E_{k-1} \to E_k \to \cdots \to A$$

  - When do we have that $[x] \in K_0(E_k)$ persists to induce a (non-trivial) class in $K_0(A)$?
  - Can we extract invariants of A from the invariants $\{\mathcal{V}(E_k, k)\}_k$?
  - Relation to quantitative K-theory [Oyono-Oyono–Yu] and NF/CPC*-systems [Blackadar–Kirchberg, Courtney–Winter]